Refinements of Perfect Equilibria

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Abstract

In this paper, we examine relations among various refinements of a perfect equilibrium. First, we compare two refinements of a perfect equilibrium, a strictly perfect equilibrium (SPE) and a truly perfect equilibrium (TPE). We show that true perfectness implies strict perfectness. We also introduce the concept of a restrictive perfect equilibrium allowing only some or all sequences of totally mixed strategies satisfying a certain property. A proper equilibrium is an example of a restrictive perfect equilibrium. Then, true perfectness imposing the most severe robustness against perturbations implies any restrictive perfectness. This proves that every truly perfect equilibrium is proper.

Key Words: refinement, perfect equilibrium, strictly perfect equilibrium, restrictive perfect equilibrium, truly perfect equilibrium

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1 Introduction

Nash equilibrium is the most important solution concept in game theory, but many games have multiple Nash equilibria and some of the equilibria may not be sensible. Selten (1975)

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refined the concept of Nash equilibrium by introducing the concept of perfect equilibrium which is stable against slight perturbations in the equilibrium strategies. The idea of small perturbations (trembling) was succeeded by many others, and enabled alternative solution concepts, among others, proper equilibrium of Myerson (1978) and sequential equilibrium by Kreps and Wilson (1982).

One dissatisfaction with those solution concepts based on perturbations is, however, that trembling is arbitrary in the sense that the refined concepts require the existence of *some* sequence of totally mixed strategies converging to the original Nash equilibrium but not *all* sequences. Although the reasonableness of a certain Nash equilibrium crucially depends upon which sequence to take, most of the solution concepts remain silent on the choice of a sequence by selectively choosing a sequence which supports the equilibrium with ignoring all the negative information from unfavorable sequences that do not support the equilibrium.

To resolve the arbitrariness, Okada (1981) proposed the concept of strictly perfect equilibrium (SPE) which is stable against all slight perturbations of strategies. Although this concept has not been widely used presumably due to the nonexistence problem, it has some interesting properties. First, it is trivial to see that every strictly perfect equilibrium is a perfect equilibrium. Also, van Damme (1983) showed that an essential equilibrium is strictly perfect.[1] It is also known that the strictly perfect equilibrium and the proper equilibrium, two refinements of the perfect equilibrium, have no inclusion relation with each other. In his book, van Damme (1987) raised this issue and instead showed that a strictly proper equilibrium which is a refinement of a strictly perfect equilibrium is proper. Vermeulen and Jansen (1996) provided a counterexample to show that a strictly perfect equilibrium need not be proper, and a counterexample of the converse that a proper equilibrium need not be strictly perfect is provided in González-Diás et al. (2010) (which was borrowed from van Damme (1987)).

The concept of the strictly perfect equilibrium can be roughly defined as follows; A Nash equilibrium σ^* is strictly perfect if for given $\epsilon > 0$ and for an arbitrary sequence $\{\eta^k\}$ such that $0 < \eta^k(s_i) < \epsilon$ for all i, there exists a totally mixed strategy equilibrium σ^k of the perturbed game for all k such that $\sigma^k \to \sigma^*$ as $\epsilon \to 0$. In this paper, we will argue that the concept of the

[1]The concept of the essential equilibrium which is stable against any arbitrary perturbations in payoffs of players was introduced by Wu and Jiang (1962).

strictly perfect equilibrium is too weak in the sense that it only requires the existence of *some* sequence of totally mixed strategy equilibria of the perturbed game that converges to the original Nash equilibrium. It requires an *arbitrary* sequence of perturbations but only *some* sequence of equilibria in the perturbed games. Accordingly, any restriction on the sequence of equilibria in the perturbed games refines the set of strictly perfect equilibria. Van Damme (1987) proposed one such solution concept, which he calls strictly proper equilibrium by imposing a restriction that the map of ϵ into an ϵ-perfect equilibrium is continuous. Kohlberg (1981) independently proposed a similar solution concept which requires the equilibrium to be the best response to any sequence of totally mixed strategies converging to it. However, we show that the truly perfect equilibrium is a stronger concept than the strictly perfect equilibrium. As such, if a strictly perfect equilibrium (SPE) is replaced by either strictly proper equilibrium or truly perfect equilibrium, the counterexample of Vermeulen and Jansen (1996) does not hold in the sense that the proposed equilibrium is neither truly perfect nor strictly proper, although it is SPE. This is because it is based on the construction of a *particular* sequence of totally mixed strategy equilibria in perturbed games to show that the proposed equilibrium is SPE.

A perfect equilibrium was defined by two alternative but equivalent ways by Selten (1965). One is a limit of a sequence of ϵ-perfect equilibria, and the other is a limit of a non-equilibrium sequence. A proper equilibrium which was introduced by Myerson (1978) was originally defined by a limit of ϵ-proper equilibria. However, properness can be also defined by a limit of a non-equilibrium sequence. We provide an alternative definition for properness in terms on a non-equilibrium sequence (satisfying some property) and show that the two definitions for properness are equivalent. In fact, this alternative definition turns out to be very useful in proving many inclusion relations among various perfectness-based solution concepts. A truly perfect equilibrium is a limit of *any unconstrained* sequence of totally mixed strategies, while a proper equilibrium is a limit of *some constrained* sequence. We will call a limit of some or any constrained sequence of totally mixed strategies *restrictive perfect equilibrium.* It is not difficult to see that the truly perfect equilibrium is a stronger concept than any restrictive perfect equilibrium, since it is harder for all sequences to satisfy some condition (being a best responses to the proposed equilibrium) than for some or all sequences with some property to satisfy the condition. Since a proper equilibrium is a restrictive perfect equilibrium, it directly follows that every truly perfect equilibrium is proper.

The paper is organized as follows. A motivating example is presented in Section 2. In Section 3, we provide the formal definition of a strictly perfect equilibrium (SPE) by Okada and of a truly perfect equilibrium (TPE) by Kohlberg (1981). In Section 4, we prove some properties of various solution concepts based on perfectness. Concluding remarks follow in Section 5.

2 Motivating Example

We consider the counterexample provided by Vermeulen and Jansen (1996) to show that a strictly perfect equilibrium need not be proper. The example is drawn in Table 1. Vermeulen and Jansen claim that (s_2, t_4) is strictly perfect but not proper. We will show that this equilibrium fails to satisfy the definition of TPE.

Consider a sequence of totally mixed strategies $\sigma_1^k = (\epsilon_1^k, 1 - \epsilon_1^k - \epsilon_3^k, \epsilon_3^k)$ where $\epsilon_1^k \gg \epsilon_3^k$. This sequence clearly converge to s_2. The expected payoffs of player 2 in the perturbed games are

$$\begin{aligned} E[\pi_2(\sigma_1^k, t_1)] &= 3\epsilon_1^k \\ E[\pi_2(\sigma_1^k, t_2)] &= 3\epsilon_3^k \\ E[\pi_2(\sigma_1^k, t_3)] &= 2(\epsilon_1^k + \epsilon_3^k) \\ E[\pi_2(\sigma_1^k, t_4)] &= 2(\epsilon_1^k + \epsilon_3^k). \end{aligned}$$

Since $\epsilon_1^k \gg \epsilon_3^k$, the best response of player 2 to σ_1^k is t_1. This implies that (s_2, t_4) is not TPE, because t_4 is not a best response to σ_1^k for $(\epsilon_1^k, \sigma_3^k)$ such that $\epsilon_1^k \gg \epsilon_3^k$.

The reason why (s_2, t_4) can be an SPE goes as follows. To player 2, t_4 can be a best response only if player 1 chooses s_1 and s_3 with similar probabilities. It is not a best response to some mixed strategy of player 1 if player 1 chooses either s_1 or s_3 almost for sure; in that case, player 2 will get almost 3 by choosing either t_1 or t_2 while he gets almost 2 by choosing t_4. To avoid this possibility, Vermeulen and Jansen picked a sequence of mixed strategies of player 1, $\sigma_1^k = (\epsilon_1^k, 1 - 3\epsilon_1^k, 2\epsilon_1^k)$ for perturbations $\epsilon^k = (\epsilon_1^k, \epsilon_2^k, \epsilon_3^k)$ such that $\epsilon_1^k > \epsilon_3^k$ and $\sigma^k = (2\epsilon_3^k, 1 - 3\epsilon_3^k, \epsilon_3^k)$ such that $\epsilon_1^k < \epsilon_3^k$. Note that if a sequence of perturbations converging to 0 shifts from one regime $(\epsilon_1^k > \epsilon_3^k)$ to another $(\epsilon_1^k < \epsilon_3^k)$, σ^k moves discontinuously with respect to a change in $(\epsilon_1^k, \epsilon_3^k)$. Van Damme (1987) proposed a strict proper equilibrium instead of TPE by imposing the requirement of continuity of the Nash sequence with respect to perturbations.

The reason why (s_2, t_4) cannot be a proper equilibrium is clearer. Player 1's choice of s_2 given t_4 can be justifiable only by high possibilities that player 2 makes mistakes of playing t_1 or t_2. However, player 2 makes more mistakes of playing t_3 because t_3 is a better response to player 1's choice of s_2 when the probabilities that player 1 makes mistakes of s_1 and s_3 are similar. Since s_2 is not a best response to such a sequence of mixed strategies, (s_2, t_4) is not TPE nor proper. This motivates us to reexamine the concept of TPE by Kohlberg (1981) and compare it with other related concepts based on perfection.

3 Preliminaries

Consider a finite strategic-form game $G = (N, S, \pi)$ where $N = \{1, 2, \ldots, n\}$, $S = \prod_{i=1}^n S_i$ and $\pi(s) = (\pi_1(s), \ldots, \pi_n(s))$. A perturbation in G is a vector $\eta = (\eta_1, \eta_2, \ldots, \eta_n)$ such that, (i) for each $i \in N$, $\eta_i = (\eta_i(s_i))_{s_i \in S_i}$ satisfies $\eta_i(s_i) > 0$ for all $s_i \in S_i$ and (ii) $\sum_{s_i \in S_i} \eta_i(s_i) < 1$. Then, for a given scalar $\epsilon > 0$, an ϵ-perturbed game of G, which will be denoted by $G(\epsilon)$, can be defined by $G(\epsilon) = (N, \Sigma(\epsilon), \pi)$ where $\Sigma_i(\epsilon) = \{\sigma_i \in \Sigma_i \mid \sigma_i(s_i) \geq \eta_i(s_i), \forall s_i \in S_i\}$ for some $\eta_i(s_i)$ such that $0 < \eta_i(s_i) \leq \epsilon$ and $\Sigma(\epsilon) = \prod_{i \in N} \Sigma_i(\epsilon)$.

Selten (1965) defines a perfect equilibrium by a limit of Nash equilibria in a sequence of perturbed games. Formally, a perfect equilibrium is defined by a limit of some sequence of ϵ-constrained equilibria as $\epsilon \to 0$, where an ϵ-constrained equilibrium for a given $\epsilon > 0$ is a Nash equilibrium of a perturbed game $G(\epsilon)$ for *some* perturbation η such that $\eta_i(s_i) \in (0, \epsilon)$ for all $s_i \in S_i$ and for all $i \in N$.[2] Similarly, Myerson (1981) defines an ϵ-perfect equilibrium and give an alternative definition of a perfect equilibrium as a limit of ϵ-perfect equilibrium.

Definition 1 (Myerson) *A totally mixed strategy profile σ^ϵ is an ϵ-perfect equilibrium if and only if for any $s_i, s_i' \in S_i$,*

$$\pi_i(s_i, \sigma_{-i}^\epsilon) < \pi_i(s_i', \sigma_i^\epsilon) \Longrightarrow \sigma_i^\epsilon(s_i) \leq \epsilon. \tag{1}$$

[2] We borrow this term from Fudenberg and Tirole (1991).

Due to Selten (1965) and Myerson (1978), we have three alternative definitions of perfectness.

Theorem 1 (Selten, Myerson) *The followings are equivalent; (i) σ is a perfect equilibrium, i.e., $\sigma = \lim_{\epsilon \to 0} \sigma^\epsilon$ where σ^ϵ is an ϵ-constrained equilibrium. (ii) There exists a sequence of totally mixed strategies $\{\sigma^k\}$ such that $\sigma^k \to \sigma$ and σ_i is a best response to σ^k_{-i} for all k. (iii) $\sigma = \lim_{\epsilon \to 0} \sigma^\epsilon$ where σ^ϵ is an ϵ-perfect equilibrium.*

The proof of equivalence is omitted, since it can be found in standard textbooks on game theory. See, for example, Fudenberg and Tirole (1991).

We now introduce two closely related refinements of a perfect equilibrium, strict perfectness by Okada (1981) and true perfectness by Kohlberg (1981).

Definition 2 (Okada) *A Nash equilibrium $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$ for G is a strictly perfect equilibrium of G if for any arbitrary sequence of perturbations $\eta^k = (\eta_1^k, \ldots, \eta_n^k)\}_{k=0}^{\infty}$ such that $0 < \eta_i^k < \epsilon^k$ and $\epsilon^k \to 0$ as $k \to \infty$, there exists a sequence of totally mixed strategy Nash equilibria σ^k of the game $G(\epsilon^k)$ such that $\sigma^k \to \sigma^*$ as $k \to \infty$.*

Roughly speaking, strict perfectness requires an equilibrium to be a limit of Nash equilibria in all sequences of games perturbed in a neighborhood of the equilibrium. It is clear that the concept of the strong strictly perfect equilibrium (SPE) is stronger than the concept of the perfect equilibrium (PE) because SPE requires a sequence of Nash equilibria for *all* perturbations while PE requires a sequence of Nash equilibria for *some* perturbations.

Definition 3 (Kohlberg) *A Nash equilibrium $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$ for G is a truly perfect equilibrium of G if for any sequence of totally mixed strategy profiles σ^k such that $\sigma^k \to \sigma^*$, $\pi_i(\sigma_i^*, \sigma_{-i}^k) \geq \pi_i(s_i, \sigma_{-i}^k)$ for all $s_i \in S_i$ and for all $i \in N$.*

Kohlberg (1981) introduced a similar notion of so-called truly perfect equilibrium which requires a best response to every sequence of totally mixed strategies converging to it. It is also clear that the concept of the truly perfect equilibrium (TPE) is stronger than the concept of the perfect equilibrium (PE) because TPE requires *all* sequences of mixed strategies converging to it while PE requires *some* sequence of mixed strategies converging to it.

In the next section, we will examine the relation among the three solution concepts.

4 Results

Some may wonder whether SPE and TPE are equivalent just as the two definitions of a perfect equilibrium are equivalent. However, it turns out that the equivalence result of the *some* sequence version is not carried over to the *all* sequence version. That is, we can show that every truly perfect equilibrium is strictly perfect, but the converse does not hold.

Theorem 2 *Every truly perfect equilibrium is strictly perfect.*

Proof. Let σ^* be a truly perfect equilibrium. For given $\epsilon > 0$ and for any arbitrary sequence of perturbations $\eta^k = (\eta_i^k)$ such that $\eta_i^k(s_i) \in (0, \epsilon)$, take a sequence of totally mixed strategies

$$\sigma_i^{\eta^k}(s_i) = \begin{cases} \eta^k(s_i) < \epsilon & \text{if } s_i \notin \text{supp}(\sigma_i^*) \\ \sigma_i^*(s_i) - \sum_{s_i \notin \text{supp}(\sigma_i^*)} \eta^k(s_i)/n_i & \text{if } s_i \in \text{supp}(\sigma_i^*) \end{cases} \tag{2}$$

where $\text{supp}(\sigma_i^*) = \{s_i \in S_i \mid \sigma_i(s_i) > 0\}$ and $n_i = |\text{supp}(\sigma_i^*)|$. By definition of TPE, σ_i^* is a best response to any sequence of totally mixed strategies σ_{-i}^k such that $\sigma_{-i}^k \to \sigma_{-i}^*$, in particular, $\sigma_{-i}^{\eta^k}$, for all $i \in N$. Let $B(\sigma_{-i})$ and $NB(\sigma_{-i})$ be the set of pure strategies of player i that are best responses (not best responses, *resp.*) to σ_i. Then, $\text{supp}(\sigma_i^*) \subset B(\sigma_{-i}^{\eta^k})$ by the above best-response argument. Note that $\sigma_i^{\eta^k}(s_i) < \epsilon$ for any $s_i \in NB(\sigma_{-i}^{\eta^k}) \subset S_i \backslash \text{supp}(\sigma_i^*)$. Therefore, σ^{η^k} is an ϵ-perfect equilibrium by Definition 1. Since $\frac{\sum_{s_i \notin \text{supp}(\sigma_i^*)} \eta^k(s_i)}{n_i} < \frac{|S_i| - n_i}{n_i}\epsilon \to 0$ as $\epsilon \to 0$, $\sigma^* = \lim_{\epsilon \to 0} \sigma^{\eta^k}$. Since σ^* is perfect for any perturbation η^k, it is strictly perfect. $\|$

The example of Vermeulen and Jansen (1996) which was provided in Section 2 can be a counterexample of the converse claim. In the example, (s_2, t_4) is a strictly perfect equilibrium, but not truly perfect as we already argued.

As is well known, Myerson (1978) introduced the concept of the proper equilibrium which is a refinement of the perfect equilibrium. Roughly speaking, it requires perturbations (mistakes) to satisfy an additional requirement such that a more costly mistake will occur with a probability of smaller order than the probability of a less costly one.

Definition 4 *For a fixed scalar $\epsilon > 0$, a totally mixed strategy profile σ^ϵ is an ϵ-proper equilibrium of a game G if for $s_i, s_i' \in S_i$,*

$$\pi_i(s_i, \sigma_{-i}^\epsilon) < \pi_i(s_i', \sigma_i^\epsilon) \Longrightarrow \sigma_i^\epsilon(s_i) \leq \epsilon \sigma_i^\epsilon(s_i'). \tag{3}$$

A proper equilibrium of G is any limit of σ^ϵ*, i.e.,* $\sigma^* = \lim_{\epsilon\to 0}\sigma^\epsilon$*.*

A limit of any sequence of ϵ-proper equilibria as $\epsilon \to 0$ is proper. Note that this definition of properness does not explicitly specify the Nash equilibrium condition of the ϵ-perturbed game unlike the definition of perfectness. This is because the Nash concept is embedded in the requirement of lower-order mistakes for better responses. As the definition of perfectness, the definition of properness can be stated in terms of a sequence of totally mixed strategies.

Theorem 3 σ^* *is a proper equilibrium if and only if there exists a sequence of totally mixed strategies* $\{\sigma^k\}$ *such that* $\sigma^k \to \sigma^*$*,* σ_i^* *is a best response to* σ_{-i}^k *for all* k *and the sequence* $\{\sigma^k\}$ *satisfies the property [P] where, for* $s_i, s_i' \in S_i$ *and for any* $i \in N$*,*

$$\pi_i(s_i, \sigma_{-i}^k) < \pi_i(s_i', \sigma_{-i}^k) \Longrightarrow \sigma_i^k(s_i) \leq \epsilon^k \sigma_i^k(s_i'), \text{ for some } \epsilon^k > 0 \text{ such that } \epsilon^k \to 0. \qquad [P]$$

Proof. ($\Longrightarrow$) Let σ^{ϵ^k} be an ϵ^k-proper equilibrium for a sequence $\epsilon^k \to 0$ and take $\sigma^k = \sigma^{\epsilon^k}$. Since $\{\sigma^k\}$ clearly satisfies the property [P] and $\sigma^k \to \sigma^*$, it remains to show that σ_i is a best response to σ_{-i}^k for all k. For $s_i \in \text{supp}(\sigma_i^*)$, we know that $\sigma^k(s_i) \to \sigma^*(s_i) > 0$, so that there exists $d > 0$ such that $\sigma^k(s_i) \geq d$ for any $k \geq k_1$ for some large k_1. Since $\epsilon^k \to 0$, there exists large k_2 such that $d > \epsilon^k$ for all $k \geq k_2$. Take $\bar{k} = \max\{k_1, k_2\}$. Then, for all $s_i \in \text{supp}(\sigma_i^*)$, $\sigma^k(s_i) > \epsilon^k$ for all $k \geq \bar{k}$. Since σ^k is ϵ^k-perfect, s_i is a best response to σ_{-i}^k for all $k \geq \bar{k}$ by Definition 1. Therefore, σ^* is a best response to σ_{-i}^k for all $k \geq \bar{k}$.

($\Longleftarrow$) Take $\{\epsilon^k\}$ and $\{\sigma^k\}$ that satisfy [P], $\epsilon^k \to 0$ and $\sigma^k \to \sigma^*$. Then, for any k, σ^k satisfies [P] so that

$$\pi_i(s_i, \sigma_{-i}^k) < \pi_i(s_i', \sigma_i^k) \Longrightarrow \sigma_i^k(s_i) \leq \epsilon^k \sigma_i^k(s_i'),$$

implying that σ^k is an ϵ^k-proper equilibrium. Therefore, $\sigma^* = \lim_{\epsilon^k\to 0}\sigma^k$ is proper. $\|$

Property [P] simply says that each player's totally mixed strategy is required to assign lower probability to the pure strategy yielding him a lower expected payoff. Due to Theorem 3, we can easily compare the set of truly perfect equilibria and the set of proper equilibria. Note that a truly perfect equilibrium must satisfy the following two conditions; (i) it is a limit of *any* sequence of totally mixed strategies that converge to it and (ii) it must be a best response to *any* such sequence, while a proper equilibrium must satisfy the conditions; (i) it is a limit of *some* sequence of totally mixed strategies that converge to it and satisfy some property [P], (ii) it must be a best response to *some* such sequence. It is obvious that it is harder for the conditions for a truly perfect equilibrium to be satisfied. Hence, a truly perfect equilibrium must be proper. This logic can be extended to any other property than [P]. For example, if we replace the condition that $\pi_i(s_i, \sigma^k_{-i}) < \pi_i(s'_i, \sigma^k_i)$ in property [P] of Theorem 3 by the condition that $\pi_i(s_i, \sigma^*_{-i}) < \pi_i(s'_i, \sigma^*_i)$, the corresponding solution concept is called *weakly proper equilibrium.* (See Definition 2.3.1 of van Damme (1991).) Notice that this condition is less restrictive than property [P], because $\pi_i(s_i, \sigma^*_{-i}) < \pi_i(s'_i, \sigma^*_i)$ implies that $\pi_i(s_i, \sigma^k_{-i}) < \pi_i(s'_i, \sigma^k_i)$ but not *vice versa*.[3] Thus, it directly follows that every proper equilibrium is weakly proper but a weakly proper equilibrium is not necessarily proper. It also follows that every weakly proper equilibrium is perfect because the perfect equilibrium does not impose any restriction on $(\sigma^k)_{k=1}^{\infty}$ such that property [P]. We will call a limit of *any* or *some* sequence of totally mixed strategies satisfying some property a *restrictive perfect equilibrium (RPE)*. A weakly proper equilibrium is also a restrictive perfect equilibrium. Then, we have the following general result.

Theorem 4 *Every truly perfect equilibrium is a restrictive perfect equilibrium (RPE).*

Proof. Obvious.

Corollary 1 *Every truly perfect equilibrium is proper.*

Now, we want to propose a new solution concept, what we call a *truly proper equilibrium*. We can define a truly proper equilibrium analogously as a truly perfect equilibrium is defined. That is, a truly proper equilibrium is defined by a limit of *any* sequence of totally mixed strategies satisfying the property [P] to which it is a best response. Formally,

Definition 5 *σ is a truly proper equilibrium if and only if for any sequence of totally mixed strategies $\{\sigma^k\}$ such that $\sigma^k \to \sigma$, σ_i is a best response to σ^k_{-i} for all k and the sequence $\{\sigma^k\}$ satisfies the property P where, for $s_i, s'_i \in S_i$ and for any $i \in N$,*

$$\pi_i(s_i, \sigma^k_{-i}) < \pi_i(s'_i, \sigma^k_i) \Longrightarrow \sigma^k_i(s_i) \le \epsilon^k \sigma^k_i(s'_i), \text{ for some } \epsilon^k > 0 \text{ such that } \epsilon^k \to 0. \qquad [P]$$

Since a truly proper equilibrium is also a restrictive perfect equilibrium, we have

[3] Since $\lim_{k\to\infty} \sigma^k_i = \sigma^*_i$, $\pi_i(s_i, \sigma^k_{-i}) < \pi_i(s'_i, \sigma^k_i)$ implies that $\pi_i(s_i, \sigma^*_{-i}) \le \pi_i(s'_i, \sigma^*_i)$.

Corollary 2 *Every truly perfect equilibrium is truly proper.*

Note that a truly perfect equilibrium implies a truly proper equilibrium, whereas a proper equilibrium implies a perfect equilibrium, since the conditions for a truly perfect equilibrium is more stringent than those for a truly proper equilibrium.

Some may suspect that (s_2, t_4) in the example provided in Table 1 may be truly proper, even if it is not truly perfect. To check this, pick any one sequence satisfying [P] among the whole set of sequences of totally mixed strategies that converge to (s_2, t_4), $(\sigma_1^k, \sigma_2^k) = ((\epsilon_1^k, 1 - \epsilon_1^k - \epsilon_3^k, \epsilon_3^k), (\delta_1^k, \delta_2^k, \delta_3^k, 1 - \delta))$ where $\delta = \sum_{i=1}^3 \delta_i^k$. Given this strategy combination, the expected payoffs of player 1 are computed as

$$\begin{aligned} E[\pi_1(s_1, \sigma_2^k)] &= \delta_1^k \\ E[\pi_1(s_2, \sigma_2^k)] &= 7(\delta_1 + \delta_2) - 3\delta_3 \\ E[\pi_1(s_3, \sigma_2^k)] &= \delta_2^k. \end{aligned}$$

Suppose $\delta_1 \gg \delta_2$. Then, $\pi_1(s_2, \sigma_2^k) > \pi_1(s_1, \sigma_2^k) > \pi_1(s_3, \sigma_2^k)$. Thus, property [P] requires that $\sigma_1^k(s_3) < \epsilon^k \sigma_1^k(s_1) < (\epsilon^k)^2 \sigma_1^k(s_2)$, implying that $\epsilon_1^k \gg \epsilon_3^k$. This in turn implies that t_1 (not t_4) would be the best response of player 2 to this perturbed mixed strategies. This concludes that (s_2, t_4) is not truly proper, either.

It is known that the proper equilibrium always exists, but the truly perfect equilibrium need not exist, because it is a refinement of the strictly perfect equilibrium the existence of which is not guaranteed. (See Figure 1.5.5 of van Damme (1991) for an example of the non-existence.) Then, could we recover the existence by slightly relaxing the concept of the truly perfect equilibrium to the truly proper equilibrium by allowing only the totally mixed strategies satisfying [P] property? Unfortunately, the answer is no. It is not difficult to construct an example in which the truly proper equilibrium does not exist, although the proper equilibrium exists. Consider a strategic-form game provided in Table 2. This game has the unique Nash equilibrium (s_2, t_2), and this is indeed proper. To see this, consider a sequence of totally mixed strategies that converge to $\sigma^* = (s_2, t_2)$, $\{(\sigma_1^k, \sigma_2^k)\}_{k=1}^\infty = \{(\epsilon_1^k, 1 - \epsilon_1^k - \epsilon_2^k, \epsilon_2^k), (\delta_1^k, 1 - \delta_1^k - \delta_2^k, \delta_2^k)\}_{k=1}^\infty$ where $\epsilon_i^k, \delta_i^k \in (0, 1)$, $\epsilon_i^k \to 0$ and $\delta_i^k \to 0$. The expected payoffs of player 1 are as follows;

$$\begin{aligned} E[\pi_1(s_1, \sigma_2^k)] &= 2 - \delta_1^k - \delta_2^k \\ E[\pi_1(s_2, \sigma_2^k)] &= 3\delta_1^k - 3\delta_2^k \\ E[\pi_1(s_3, \sigma_2^k)] &= 2 - \delta_1^k - \delta_2^k. \end{aligned}$$

Since player 1 is indifferent between s_1 and s_3, property [P] does not impose any restriction on ϵ_1^k and ϵ_2^k. On the other hand, the expected payoffs of player 2 can be computed as follows;

$$\begin{aligned} E[\pi_2(\sigma_1^k, t_1)] &= 2(1 - \epsilon_1^k - \epsilon_2^k) + 2\epsilon_2^k \\ E[\pi_2(\sigma_1^k, t_2)] &= \epsilon_1^k + 3(1 - \epsilon_1^k - \epsilon_2^k) \\ E[\pi_2(\sigma_1^k, t_3)] &= 3(1 - \epsilon_1^k - \epsilon_2^k) + \epsilon_2^k. \end{aligned}$$

Thus, t_2 is a best response of player 2 if $\epsilon_1^k \gg \epsilon_2^k$, while t_3 is if $\epsilon_2^k \gg \epsilon_1^k$. This implies that (s_2, t_2) is not truly proper; hence, no truly proper equilibrium in this game, since it is the unique Nash equilibrium.

5 Concluding Remarks

In this paper, we showed that every truly perfect equilibrium is strictly perfect and that every truly perfect equilibrium is proper by proving that every truly perfect equilibrium is restrictively perfect. We also proposed the concept of truly proper equilibrium and that every truly perfect equilibrium is truly proper, but it does not need to exist, either. We look forward to a less stringent solution concept than the truly perfect equilibrium and even the truly proper equilibrium that can guarantee existence.

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		II			
		t_1	t_2	t_3	t_4
	s_1	$(1,3)$	$(0,0)$	$(0,2)$	$(0,2)$
I	s_2	$(7,0)$	$(7,0)$	$(-3,0)$	$(0,0)$
	s_3	$(0,0)$	$(1,3)$	$(0,2)$	$(0,2)$

Table 1: (s_2, t_4) is SPE but not TPE

		II		
		t_1	t_2	t_3
	s_1	$(1,0)$	$(2,1)$	$(1,0)$
I	s_2	$(2,2)$	$(3,3)$	$(0,3)$
	s_3	$(1,2)$	$(2,0)$	$(1,1)$

Table 2: The truly proper equilibrium need not exist.