The Debasement Puzzle, Gresham's Law, and a Theory of Coinage

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March 12, 2014

Abstract

The debasements in medieval Europe were often followed by large minting volume and seigniorage profits, constituting the so-called debasement puzzle. In the meanwhile, old heavier coins were observed to co-circulate with new lighter coins, casting a doubt on Gresham's law. Here we offer a theory of coinage that appeals to divisibility and portability of commodity money. Coins are distinguished as different denominations by their metal contents. In a parameterized model, we find that the minting volume does surge following debasement while Gresham's Law may or may not hold.

JEL classification: E40; E42

Keywords: The debasement puzzle; Gresham's Law; Denominations; Medieval coinage; Commodity money

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1 Introduction

In the medieval Europe, debasements of coins—reduction of the weight or fineness of contained precious metal such as silver or gold in coins—are not rare. When there was a debasement of a sort of coins, a person could take old heavier-weight coins or bullion to a mint for exchange of new lighter-weight coins. The difference between metal contained in his old coins or bullion and metal contained in his new coins is seigniorage collected by the mint. By the conventional wisdom, what motivated a king or queen to order the debasement is fiscal consideration, i.e., for seigniorage collection (cf. Spufford [18]); recently Sargent and Velde [16] provide an influential monetary view, i.e., the debasement as a means to resolve shortage in small coins. Whatever motivations may be, debasements are puzzling because they actually worked, i.e., induced voluntary surrendering of metal and huge minting volumes, hence, seigniorage to mints for new coins. Indeed, Rolnick, Velde, and Weber [14] report the following debasement puzzle:

> "they [the medieval debasements] were followed by unusually large minting volumes and by increased seigniorage; old and new coins circulated concurrently; and, at least some of the time, coins were valued by weight. These facts constitute a puzzle because debasements provide no additional inducements to bring [old] coins to the mint."[1]

The medieval debasements are also the origin of Gresham's law, which says bad money drives out good money—new lighter-weight coins are bad and old heavier-weight are good.[2] Despite its renounced status in economics, Gresham's law is not without controversy for its empirical validity.[3] The

[1]According to [14], in the Great Debasement, minting activity increased by a factor of 2.8, and the Crown raised a quarter of its revenues through the mint.

[2]Fetter [3] describes the history of how a statement regarding debasement and the exchanges by Gresham in 1558 (in a letter to Queen Elizabeth) was recovered and further reformed to a fundamental and universal law in economics as Gresham's law in the nineteenth century. The law is also referred to as the Law of Oresme, Copernicus, and Gresham in recognizing the other independent medieval contributors, and it is also even related to far more ancient writing such as *The Frogs* of Aristophanes (c.f. Balch [1]).

[3]For example, Rolnick and Weber [13] document a few exceptions to Gresham's law in the nineteenth-century U.S. and seventeenth-century English with bimetallism; these exceptions are further disputed by Greenfield and Rockoff [3]. In a more recent study, Li [7] argues that Gresham's law was rather ineffective following the sixteenth-century English Great Debasement.

debasement puzzle seems suggestive for why Gresham's law may be controversial: bad money did drive out some good money (old coins were brought to the mint), but bad money did not drive out all good money.

Here we offer a theory of coinage and use it to understand the debasement puzzle and clarify validity of Gresham's law in the debasement context. Our theory starts from a simple observation: coins with different metal contents constitute a denomination structure. But why do multiple denominations matter at first place? Following Lee, Wallace, and Zhu [6], we appeal to indivisibility of money and costs to carrying monetary objects. Both factors make perfect sense for the medieval commodity-money system: coins cannot be too small in size and too low in finiteness (cf. Redish [10, pp.18-24]), and it is burdensome to carry many coins.

Formally, we adapt the model in [6] for commodity money. Coins are distinguished as different denominations by their metal contents but not their legal-tender values. New lighter coins introduced by a debasement of one sort of coins reduce the indivisibility problem, while old heavier coins bear less carrying costs. There is no obvious reason that the metal contents of old and new coins alone can determine whether people desire new coins or whether new coins drive out old coins. Intuitively, a limited supply of metal and a great degree of debasement may lead to a strong indivisibility-reducing effect and, hence, make new coins much desirable. To deliver more, the model is parameterized and solved by numerical methods. When the pre-debasement economy is in a steady state, debasement does induce a large minting value and the old and new coins do circulate concurrently for years. Even asymptotically Gresham's law is not universal.

There is a small literature that tackles the debasement puzzle. In a cash-in-advance model, Sargent and Smith [15] assume that good and bad coins circulate by tale and show that in such an equilibrium (if it exists) people bring some good coins into the mint to melt.[4] In matching models with one unit upper bound on coin holdings, Velde, Weber, and Wright [20] and Li [8] use side payments offered by mints as incentives for people to bring metal for new coins. None of these three models leaves room for new and old coins

[4]Built on [15] and adding an extra penny-in-advance constraint, Sargent and Velde [16] explain recurrent debasements as means to deal with recurrent shortage in small coins. On the empirical ground, Rolnick, Velde, and Weber [14] argue that by-tale circulation violates facts documented in the debasement puzzle, and that by-tale circulation would have induced a much larger minting volume than observed (while the minting volume following a debasement jumped, only a portion of old coins were minted).

to serve as different denominations. Each of these models does predict that Gresham's law is not universal and gives a distinct reason for the law to hold at some parameter space—asymmetric information in [20], the government transaction policy in [8], and the circulation-by-tale assumption in [15].

2 The basic model

To ease exposition, the basic model only describes the pre-debasement economy.

2.1 The physical environment

The physical environment has commodity money with multiple denominations. Our formulation for commodity money follows Velde and Weber [19], which is meant to capture the idea that money stock is not constant due to hoarding, international flow, and industry use of metal.[5] Our formulation for the denomination structure follows Lee, Wallace, and Zhu [6].

Time is discrete, dated as $t \geq 0$. There is a unit measure of infinitely lived agents. There is a durable commodity, called silver. Silver is not perfectly divisible; its smallest unit is normalized as unity. Silver has monetary and non-monetary uses. For non monetary use, silver can be costlessly converted into and back from a product, called jewelry, whose smallest unit is an integer η; jewelry is measured by its silver content. For monetary use, there is a mint which makes coins by silver. There are K types of coins and we let $m = (m_1, , ..., m_K)$ represent the denomination structure, where m_k is silver content in a type-k coin and $m_k > m_{k-1}$. There is an exogenous (integer) bound on an agent's silver wealth, denoted B. Silver has a fixed stock M and the start-of-date-0 distribution of silver among agents, denoted π_0, is public information.

Letting $\mathbf{Y} = \prod_{k=1}^{K}\{0, 1, ..., B/m_k\}$ and $\mathbf{X} = \{0, 1, ..., B/\eta\}$, then $(y, x) \in \mathbf{Y} \times \mathbf{X}$ denotes an agent's generic portfolio of wealth in silver, where $y = (y_1, y_2, ..., y_K)$ is his holdings of coins (he holds y_k units of type-k coins) and x is his holding of jewelry. When an agent with the portfolio (y, x) meets the mint (at the timing specified below), he can play a lottery $\sigma \in \Gamma(y, x)$ with

[5] For the same purpose, we may alternatively adopt the formulation in Sargent and Wallace [17]. Both of these two formulations differ from the convention in matching models that treats commodity money as Lucas trees.

the mint, where

$$\Gamma(y,x) = \{\sigma : \textstyle\sum_{(y',x')\in \mathbf{Y}\times\mathbf{X}} (m\cdot y' + \eta x')\sigma(y',x') = m\cdot y + \eta x\} \tag{1}$$

is a set of probability measures on $\mathbf{Y}\times\mathbf{X}$; by playing the lottery, the agent's portfolio becomes (y', x') with the probability $\sigma(y', x')$ and pays the minting costs for coins (in the form of disutility) as specified below.[6]

Each date consists of two stages, 1 and 2. For an agent who starts the date with the portfolio (y, x), he can play a lottery $\sigma \in \Gamma(y, x)$ with the mint at stage 1. At stage 2, agents head into a decentralized market where they are randomly matched in pairs. Jewelry cannot be carried into the market while coins can. There is a cost to carrying coins (in the form of disutility) as specified below. In each pairwise meeting, with equal chance one agent becomes a buyer and another becomes a seller. The seller can produce a perishable good, called rice, which is only consumed by the buyer. Trading histories are private so that credits between the two agents are ruled out and, in particular, the buyer must pay the seller some coins for exchange of rice. In their coins-for-rice trade, the buyer makes a take-it-or-leave-it offer to the seller. Each agent's wealth portfolio is observed by his meeting partner.

If an agent starts with (y, x) and ends with (y', x') at stage 1, and if he consumes $q_b \geq 0$ of rice (when he is a buyer) and produces $q_s \geq 0$ (when he is a seller) at stage 2, then his realized utility in that date is

$$u(q_b) - q_s + \upsilon(\eta x') - \textstyle\sum_k \phi_k \max\{y'_k - y_k, 0\} - \sum_k \gamma_k y'_k,$$

where $\phi_k > 0$ and $\gamma_k > 0$ are the unit minting cost and carrying cost for type-k coins, respectively. By the formulation, the agent does not get direct utility from jewelry, coins, or pure silver in stage 1, and he gets direct utility from jewelry but no coins or pure silver at stage 2. The utility functions u and υ satisfy $u', \upsilon' > 0$, $u'', \upsilon'' < 0$, $\upsilon(0) = u(0) = 0$, and $u'(0) = \infty$. The agent maximizes expected discounted utility with discount factor $\beta \in (0, 1)$.

[6]In (1) and below, $a \cdot b$ denotes the inner produce of vectors a and b. The set of lotteries in (1) differs from its counterpart in [6]. A lottery in the latter preserves the individual (nominal) wealth in each portfolio realization. As we do not follow [6] to assume that m_k/m_1 is an integer, it is too restrictive to let lotteries preserve the individual wealth in silver. For example, if $K = 2$, $m_2/m_1 = 1.4$, and one's start-of-stage-1 portfolio is $(y_1, y_2, x) = (0, 1, 0.5)$, then such lotteries give him little room to change his portfolio. Technically, lotteries satisfying (1) also imply concavity of some value functions.

2.2 Equilibrium

To define equilibrium, let π_t and θ_t denote two probability measures on $\mathbf{Y}\times\mathbf{X}$, where $\pi_t(y,x)$ and $\theta_t(y,x)$ are the fractions of agents with the wealth portfolio (y,x) at the start and at the end of date-t stage 1, respectively. Let w_t and h_t denote two expected discounted utility function on $\mathbf{Y}\times\mathbf{X}$, where $w_t(y,x)$ and $h_t(y,x)$ pertain to expected discounted utility for an agent whose portfolio is (y,x) at the start and at the end of date-t stage 1, respectively.

In terms of h_t, the stage-1 portfolio-choice problem for an agent who holds (y,x) at the start of date-t stage 1 can be expressed as

$$\begin{aligned} & g(y,x,h_t) \\ & = \max_{\sigma\in\Gamma(y,x)} \textstyle\sum_{(y',x')}[h_t(y',x')+\upsilon(\eta x')-\sum_k \phi_k \max\{y'_k-y_k,0\}]\sigma(y',x'). \end{aligned} \tag{2}$$

Let $\Delta_1[y,x,h_t]$ be the set of maximizers for the problem in (2). In terms of w_{t+1}, the trade in a pairwise meeting between a buyer with (y_b,x_b) and a seller with (y_s,x_s) can be described as follows. Let

$$f(y_b,x_b,y_s,x_s,w_{t+1}) = \max_{(q,l)}[u(q)+\beta w_{t+1}(y_b-\iota,x_b)] \tag{3}$$

subject to

$$-q+\beta w_{t+1}(y_s+\iota,x_s)\geq \beta w_{t+1}(y_s,x_s), \iota\in L(y_b,y_s). \tag{4}$$

In (4), $L(y_b,y_s)$ is the set of feasible coins transfers from the buyer to the seller, that is,

$$\begin{aligned} L(y_b,y_s) = \{ & \iota\in\mathbf{Y}:\iota=\iota_b-\iota_s, \iota_b,\iota_s\in\mathbf{Y}, \\ & \iota_{b,k}\leq y_{b,k}, \iota_{s,k}\leq y_{s,k}\}. \end{aligned} \tag{5}$$

Let $\Delta_2[y_b,x_b,y_s,x_s,w_{t+1}]$ denote the set of measures that represent all randomizations over the optimal transfers of coins for the maximization problem in (3).

Given h_t, the function w_t satisfies

$$w_t(y,x) = g(y,x,h_t). \tag{6}$$

Given w_{t+1} and θ_t, the function h_t satisfies

$$h_t(y,x) = 0.5\beta w_{t+1}(y,x) + 0.5\textstyle\sum_{(y',x')}\theta_t(y',x')f(y,x,y',x',w_{t+1}) - \sum_k \gamma_k y_k. \tag{7}$$

Given π_t, the measure θ_t satisfies

$$\theta_t(y,x) = \sum\nolimits_{(y',x')} \pi_t(y',x')\sigma(y,x) \tag{8}$$

for some $\sigma \in \Delta_1[y,x,h_t]$. Given θ_t, the measure π_{t+1} satisfies

$$\pi_{t+1}(y,x) = \sum\nolimits_{(y_b,x_b,y_s,x_s)} \theta_t(y_b,x)\theta_t(y_s,x_s)[\delta(y,x) + \delta(y_b - y + y_s, x)] \tag{9}$$

for some $\delta \in \Delta_2[y_b,x_b,y_s,x_s,w_{t+1}]$, where $\delta(y,x)$ is the proportion of buyers with (y_b,x_b) who leave with (y,x) after meeting sellers with (y_s,x_s).

Definition 1 *Given π_0, a sequence $\{w_t,\theta_t,\pi_{t+1}\}_{t=0}^{\infty}$ is an equilibrium in the pre-debasement economy if it satisfies (2)-(9) with $\Gamma(y,x)$ given by (1). An equilibrium is a monetary equilibrium if $\eta\sum_x x\cdot\theta_t(0,x) < M$ for some t. A triple (w,θ,π) is a steady state if $\{w_t,\theta_t,\pi_{t+1}\}_{t=0}^{\infty}$ with $w_t = w$, $\theta_t = \theta$, and $\pi_t = \pi$ for all t is an equilibrium.*

3 Debasement

To keep exposition simple, we consider the situation that only one of the existing K types of coins, say, type j, is debased. That is, the mint provides type-j coins only with debased silver content and it remains to provide the other K-1 types coins with their pre-debasement silver contents. While the mint does not provide type-j coins with pre-debasement silver content any more, the existing non-debased type-j coins can still be hold by agents. To keep consistency of notation, let the vector $(m_1,,...,m_K)$ denote silver contents in types of coins that are provided by the mint after debasement such that $m_k > m_{k-1}$, let m_o denote silver content in a pre-debasement type-j coin, and let $m = (m_o,m_1,,...,m_K)$. Letting $\mathbf{Y} = \{0,1,...,B/m_o\} \times \prod_{k=1}^{K}\{0,1,...,B/m_k\}$ with a generic element $y = (y_o,y_1,y_2,...,y_K)$, then when an agent with portfolio $(y,x) \in \mathbf{Y}\times\mathbf{X}$ meets the mint, the set of lotteries he plays with the mint is

$$\Gamma(y,x) = \{\sigma : \sum\nolimits_{(y',x')\in\mathbf{Y}\times\mathbf{X}} (m\cdot y' + \eta x')\sigma(y',x') = m\cdot y + \eta x, \quad \sigma\{(y',x') \in \mathbf{Y}\times\mathbf{X} : y'_o \le y_o\}\}. \tag{10}$$

Compared with the set of lotteries in (1), the set of lotteries in (10) has an additional constraint $y'_o \le y_o$, saying that the mint does not provide old coins any more. With $\Gamma(y, x)$ defined in (10), all other descriptions of the pre-debasement environment and equilibrium are carried over without any change. In particular, we have the following definition parallel to Definition 1.

Definition 2 *Given π_0, a sequence $\{w_t, \theta_t, \pi_{t+1}\}_{t=0}^{\infty}$ is an equilibrium in the pre-debasement economy if it satisfies (2)-(9) with $\Gamma(y, x)$ given in (10). An equilibrium is a monetary equilibrium if $\sum_x \theta_t(0, x) < M$ for some t. A triple (w, θ, π) is a steady state if $\{w_t, \theta_t, \pi_{t+1}\}_{t=0}^{\infty}$ with $w_t = w$, $\theta_t = \theta$, and $\pi_t = \pi$ for all t is an equilibrium.*

4 Existence results

For existence of a monetary equilibrium, from now on we maintain a simple sufficient condition

$$0.5\frac{B-M}{B}u[\frac{\beta}{1-\beta}\left(v(B)-v(B-m_1)\right)] \geq (1+\frac{0.5\beta}{1-\beta})[v(B)-v(B-m_1)]+\phi_1+\gamma_1. \tag{11}$$

What (11) requires is that the upper bound on silver wealth is not too strict, the smallest coin is not too large, and the minting and carrying costs of the smallest coin are not too great.

Proposition 1 *(i) For any given π_0 there exists a Definition-1 monetary equilibrium $\{w_t, \theta_t, \pi_{t+1}\}_{t=0}^{\infty}$ such that w_t is concave, all t. (ii) There exists a Definition-1 monetary steady state (w, θ, π) such that w is concave.*

Proof. All proofs are in the appendix. ■

Analogously, we have the following results for the post-debasement economy.

Proposition 2 *(i) For any given π_0 there exists a Definition-2 monetary equilibrium $\{w_t, \theta_t, \pi_{t+1}\}_{t=0}^{\infty}$ such that w_t is concave, all t. (ii) There exists a Definition-2 monetary steady state (w, θ, π) such that w is concave.*

We analyze the economy's response to a debasement by two related approaches. One approach is through the long-run comparative statics; that is, we compare a Definition-1 steady state that is associated with the pre-debasement economy and a Definition-2 steady state that is associated with the post-debasement economy. Another approach is through the direct dynamic process; that is, we let the pre-debasement economy be in a Definition-1 steady state (w, θ, π) and let the dynamic process following the debasement be a Definition-2 equilibrium whose π_0 is implied by π. Both approaches are carried out numerically in the next section.

5 Response to a debasement: numerical analysis

In this section we use numerical methods to carry out the two approaches indicated in the end of the last section. For illustrative purposes, we use a denomination structure that simplifies the computation while preserve the important features of the model. We begin with parameterization and then turn to computational details.

5.1 Parameterization

We normalize the smallest denomination as $m_1 = 1$. For the basic model, we start with two denominations (hence $K = 2$) and set the larger denomination as $m_2 = 6$, so that there is enough space between m_1 and m_2 for us to examine the different scenarios where m_2 is debased with different magnitude.

For the total stock of silver, we set $M = 8$ so that the per capital wealth is enough to support the large coin used as money in pairwise meeting[7].

For the upper bound on silver wealth, we find that $B = 3$ works fine and making it larger will not change the result much.

For the smallest unit of jewelry η, we identify it with the most common and smallest measure of precious metal, *troy ounce*, which weighs at 31 grams. On the other hand, the per capita silver wealth in medieval ranged from 30 gram to 90 gram. So we set η close to M. In order to reduce the dimension, we increase it slightly and set $\eta = 10 > M$. As is showed later, setting $\eta = M = 8$ will not have significant effect on results.

We set the length of per period as a quarter, and hence the discount rate $\beta = 0.975$, implying an annual discount rate of 10%[8].

[7] If M is too small, m_2 will mostly be used as store of value rather than medium of exchange. On the other hand, a M too large will increase the computing burden

[8] There are a few papers (e.g. Kimball [4]) that discuss the medieval peasants' utility

	$u(q)$	$\upsilon(x)$	β	m_1	m_2	η
Main setting	$\frac{q^{1-\sigma}}{1-\sigma}$	$\epsilon\frac{x^{1-\sigma_x}}{1-\sigma_x}$	0.975	1	6	10
Alternatives				$\{0.8, 2\}$	$\{4, 8\}$	8

Table 1: Summary of model specifications and parameter choices.

We assume that the utility function for rice and jewelry take the form $u(q) = q^{1-\sigma}/(1-\sigma)$ and $\upsilon(x) = \epsilon \cdot x^{1-\sigma_x}/(1-\sigma_x)$, with $\sigma, \sigma_x \in (0,1)$ and $\epsilon \in (0,1)$. We mainly focus on the case of $\sigma = \sigma_x = 0.5$ and $\epsilon = 0.015$.

For the minting cost $(\phi_1, ..., \phi_K)$, we assume that $\phi_k = \upsilon(\zeta_k)$, where ζ_k represents the brassage (in terms of silver) for minting one unit of coin k. According to Sargent and Velde [16, pp. 50-52], the medieval mints typically charged a brassage which was concave in the size (metal content) of the coin, and we follow Lee and Wallace [5] in using square root to capture this concavity. The historical data provided by Sargent and Velde [16, pp. 51] shows that the brassage for coin with silver content roughly at 1 gram ranged from 2 mg to 9 mg, or 2% to 9%. And we take 4%, so $\zeta_k = 0.04\sqrt{m_k}$.

As for $(\gamma_1, ..., \gamma_K)$, it is difficult to directly pin down some numerical value to capture the physical inconvenience, e.g. sorting and counting, incurred by carrying the coins. However, one alternative is to re-interpret γ as the disutility associated with the loss or depreciation of coins, we have $\gamma_k = \upsilon(L \cdot m_k)$, where L is the per period loss rate. Based on the depreciation rate estimated by Patternson [9], we set $L = 0.0025$. Constructed this way, γ_k will be increasing and concave in k. A variation we shall consider in our exercises will be letting γ_k identical to all denominations, i.e. $\gamma_k = \overline{\gamma}$. This is to give prominence to large coin's advantage (few coins) while minimize its disadvantage (weight per coin). For a meaningful comparison, we set $\overline{\gamma} \equiv \upsilon(L \cdot m_1)$.

In Table 1 we summarize the parameter choices, and the alternative choices to be discussed in next sections.

discount rate. Although their estimates vary, all argue that medieval peasants had utility discount rate higher than ours. Here, our choice of a β associated with annual discount rate of 10% is in line with Lee and Wallace [5], who also focus on medieval monetary issues.

5.2 Pre-debasement and post-debasement steady states

Here we compute Proposition-1(ii) steady state and Proposition-2(ii) steady state. We cannot prove uniqueness or local stability of such a steady state. But given our choice of parameters, our algorithm converges to the steady state we find out regardless of the initial conditions. Details about the algorithm are given in the appendix.

We first look at a Proposition-1(ii) steady state with two denominations and for a comparison made below, a Proposition-1(ii) steady state with only one small denominations. In Table 2 we consider several scenarios for two denominations with variation in some parameters, and report several statistics in four categories. The basic model corresponds with the main parameter settings in table 1. Under Total Stock, we record the total stock of different coins and jewelry (in pieces). Under Pairwise Meeting, we record the total number of pairs between which non-trivial trades take place (with a total of 0.5 matched pairs per period); avg pay and avg output is the average net payment in silver and average output among these pairs. Under Circulation, we have the number of coins (in pieces) that changes hands during each period; the larger the figure, the more often the coin is used as a medium of exchange. Finally, we report the minting volume for each coin. Note that when in steady state, the melting volume equals the minting volume so as to maintain a constant aggregate portfolio.

A few observations about the two-denomination variations are now in order. First, with a uniform carrying cost, there exist more large coins than in the case of a concave γ. Second, when the silver content of the large coin m_2 is smaller, people will hold more large coins because they are more affordable and more useful in transactions, as is manifested by its increased circulation volume. The effect of m_2 on the pairwise meeting is quite small. The intuition is that the pairwise meeting is mostly determined by the small coin, which is the more commonly used and held coin. Therefore, a different m_2 will only affect a very small fraction of matched pairs. Third, when the silver content of the small coin m_1 is greater, people will hold less small coin, and more large coins. In circulation, small coin is also used less and large coin used more. Finally, note that when either m_1 or m_2 is reduced, the total stock of jewelry will become lower. This is because a smaller m_1 or m_2 means money being more divisible, and the set of denomination becomes more efficient in payments during pairwise meeting. Therefore people will be more willing to hold silver in the form of coins, rather than hoard it as

	Total Stock			Pairwise Meeting			Circulation		Mint Vol.	
Models	m_1	m_2	Jewelry	trading pairs	avg pay	avg output	m_1	m_2	m_1	m_2
Basic	3.7681	0.0619	0.3861	0.4562	1.0046	1.8509	0.4923	0.0085	0.0937	0.0797
$\gamma_k = \overline{\gamma}$	3.6851	0.0756	0.3862	0.4562	1.0042	1.8509	0.4935	0.0089	0.1018	0.0934
$m_2 = 4$	3.0952	0.2618	0.3858	0.4565	1.0032	1.8491	0.5038	0.0222	0.1085	0.0942
$m_2 = 8$	4.1128	0.0031	0.3863	0.4566	1.0052	1.8524	0.4609	0.0003	0.1275	0.0161
$m_1 = 0.8$	4.4343	0.1227	0.3717	0.4734	0.8021	1.5479	0.5989	0.0182	0.1300	0.1363
$m_1 = 2$	1.8957	0.0651	0.3818	0.4103	1.9973	2.6194	0.4279	0.0170	0.2356	0.1968

Table 2: Steady states with two denominations.

		Total Stock		
Models	m_1	old m_2	new m_2	Jewelry
Pre-debasement ($m_2 = 6$)	3.7681	0.0619	–	0.3861
Post-debasement ($m_2 = 4$)	3.0952	0	0.2618	0.3858
Post-debasement ($m_2 = 3$)	2.7547	0	0.4631	0.3856

Table 3: Steady states before and after debasing the large coins.

jewelry.

Next we look at a Proposition-1(iv) steady state following debasement. Historically, medieval debasements involved both the small denomination coins and the large denomination coins, as is documented in [16]. Unlike large denomination coins, small denomination coins are commonly held and frequently circulated and, therefore, its debasement may lead to different consequences that the debasement of the large coins. So we analyze these two kinds of debasement separately.

Debase the large denomination coins We consider two cases: (a) the large coin is debased from containing 6 units of silver to 4 units and (b) the large coin is debased from 6 to 3. Here and below, we refer to the debased new coin and the undebased old coin as "new coin" and "old coin" respectively, and the small coin (containing 1 unit of silver) as the "small coin." Table 3 displays the pre-debasement and post-debasement steady states for cases (a) and (b), respectively.

In both cases, Gresham's law holds; that is, the new coin replaces the old coin as the large denomination coin. So the post-debasement steady state is reduced to a steady state with only two denomination—the small coin and the new coin and, therefore, the different degrees of debasement matters only to the extent as different denomination structures as is discussed in Table 2. For this reason, we report only the Total Stock columns. For other characteristics of the post-debasement steady state, one can refer to the two denomination steady state in Table 2.

Debase the small denomination coins For simplicity, we assume the economy is in a one-denomination environment prior the debasement, where the only denomination is small coin with silver content $m_1 = 3$. The debasement reduces its silver content to $m_1 = 2$. The resulting steady states are showed in Table 4.

	Total Stock			Pairwise Meeting			Circulation		Mint Vol.	
Models	old m_1	new m_1	Jewelry	trading pairs	avg pay	avg output	old m_1	new m_1	old m_1	new m_1
Pre-debase	1.3855	–	0.3844	0.3666	3	2.9516	0.3666	–	0.4118	–
Pre-debase	0.8960	2.1651	0.0982	0.4377	1.1180	1.7720	0.3887	0.6534	–	0.1300

Table 4: Steady states before and after debasing the small coin from 3 to 2.

Here Gresham's law does not hold; that is, the old coins are not driven out by the new coins. There are a few reasons behind this difference between debasing the small coin and debasing the large coin. First, the old coin and new coin are more of a supplement relation than of a substitute relation, even though the new coins replaces the old coins as the major denomination. More specifically, the presence of new coins facilitates the circulation of old coins because there is more combination of coin transfers can come out of them. As a result, coins in general become more useful as medium of exchange and, hence, people tend to hold more silver in the form of coins instead of jewelry. The increase in the number of trading pairs also suggests that more trading options are available with these two types of coins.

Second, the old coins are not too large compared with the new coins, and thus are still useful in transactions. Third, it is easy for holders of old coins to switch to holding new coins, since they can simply do so during a pairwise meeting when they meet a seller with some new coins, and receive new coins as changes. They do not need to turn to the mints for such portfolio changes. Finally, the old and new coins are valued distinctively different in pairwise transaction.

5.3 The dynamic process following debasement

As already noted, we let the pre-debasement economy be in the above-computed Proposition-1(ii) steady state and the dynamic process following debasement is a Proposition-2(i) equilibrium whose initial distribution is implied by the Proposition-1(ii) steady-state distribution. We cannot prove that there is a Proposition-2(i) equilibrium that converges toward the above-computed Proposition-2(ii) steady state. Nonetheless, we design our algorithm by assuming that the computed Proposition-2(i) equilibrium does converge to the Proposition-2(ii) steady state. In our algorithm, we approximate the convergence by assuming that it occurs after T periods, where T is large enough. In computation, we find that $T = 100$ is good enough for approximation; details about the algorithm are in the appendix.

Debase the large denomination coins We consider the same two cases in the above steady-state comparison. For case (a) (the large coin is debased from 6 to 4), Figures 1 and 3 shows the total stock of each coin and jewelry and the minting volume in the dynamic process following the debasement. There are three interesting observations from those figures.

First, the number of new coins increases while the number of old coins

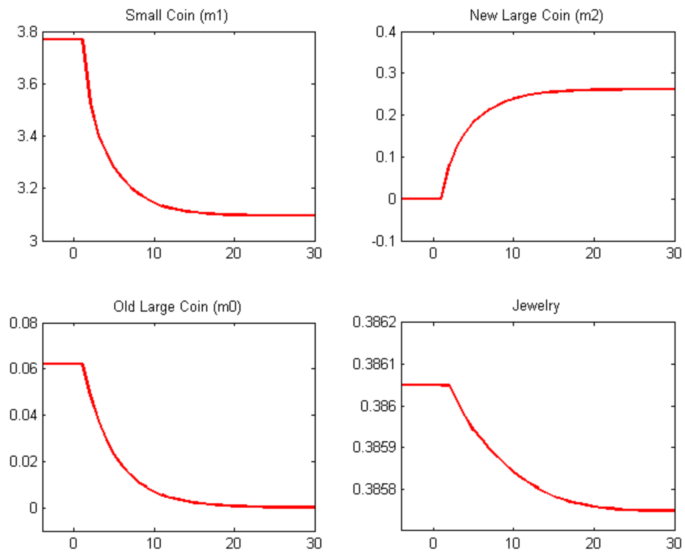

Figure 1: Response of aggregate portfolio following a debasement. Basic model.

decreases and the new coins not only take over the silver from the old coins, it also absorbs some of the silver stock from the small coins and the jewelry. That is because this new lighter coin increase the divisibility of money, induce more people to hold it.

Second, while eventually all the old coins are entirely driven out by the new coins (as discussed in the steady-state comparison), the old coins do not disappear right away after the debasement, neither is their presence in pairwise trading. Moreover, Figure 2 shows that the old coins continue to serve as a medium of exchange, alongside in pairwise transactions for some time.

Third, the total minting volumes more than doubled after the debasement. A further break down of the total minting volume suggests that the surge is mainly attributed to the coinage of the new coin. As is showed in steady states comparison, the lighter large coin is more attractive to the heavier large coin, since the former provides more divisibility. So once the debasement occurs, not only those who would have minted the old coin are now minting the new coin, but also some of those who would not have minted the old coin, since they found the new lighter large coin is more worthwhile

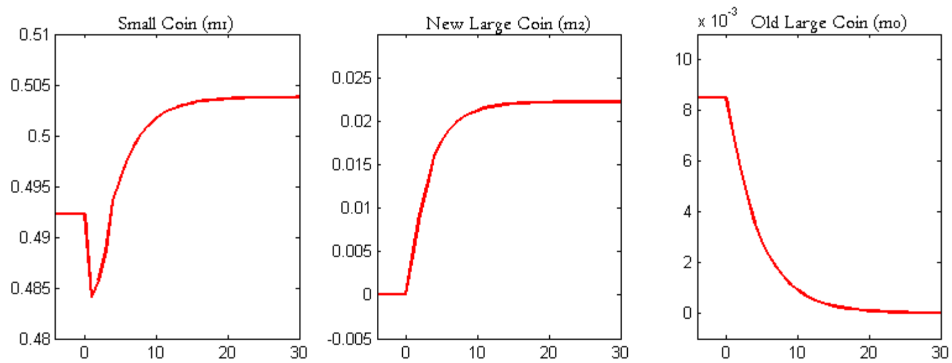

Figure 2: Number of coins circulating in pairwise trading after the debasement.

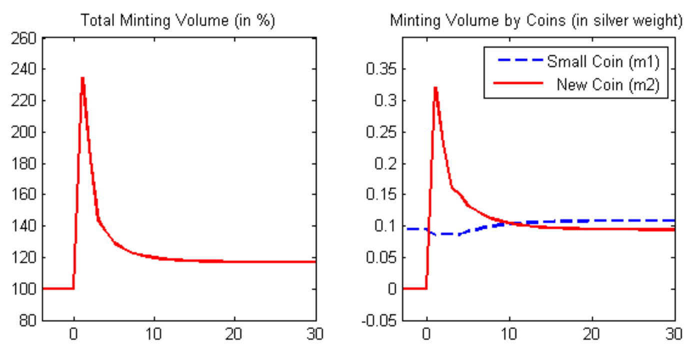

Figure 3: Minting Volume of Silver after debasement.

holding than the small coin or jewelry.

For case (b) (the large coin is debased from 6 to 3), the results are displayed in the following figure 4 and 5. All the results are qualitatively the same with the previous scenario, but quantitatively more significant. Notably, the ensuing total minting volume increases much more dramatically, by more than 450%. This is because now the new coin, weighted 3 unit of silver, provides more divisibility compared with the 4 unit new coin in previous scenario, and is more appealing so that more people would rush to the mint when it is available at the mints.

Debase the small denomination coins As in the above steady-state compassion, there is only one small denomination prior the debasement and the

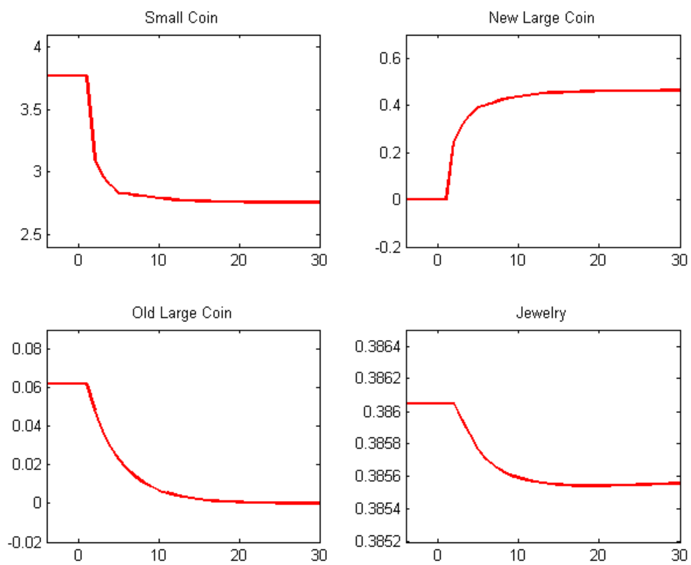

Figure 4: Stocks of different coins and jewelry after debasing the large coin from 6 unit to 3 unit.

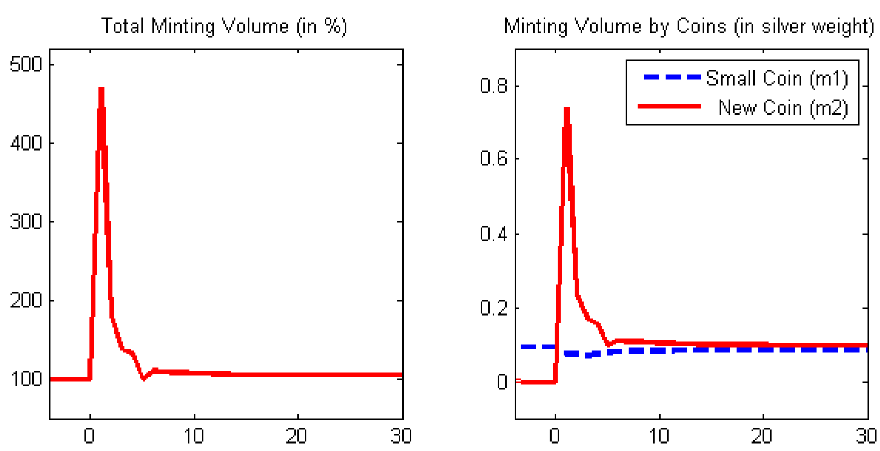

Figure 5: Minting Volume of Silver after debasing the large coin from 6 unit to 3 unit..

coin is debased from 3 to 2.[9] Figure 6 shows the dynamics after the debasement. After the debasements, the old coin circulation first take a dip and then rise. At first, the emergence of the new coins makes people with both coins unwilling to use old coins to pay for goods, since the new lighter coins are more efficient. However, as the new coins gradually become popular, people tend to use the old coins more often because they can expect sellers to give back new coins as changes. The minting volume increases by more than 650% after the debasement. Since the small coins are the major denomination used in transactions, debasement of small denomination coins increases the divisibility of money more significantly than the debasement of the large denomination coins, and therefore the minting volume surge generated is also more significant.

6 The concluding remarks

Commodity money occupies the most part of monetary history in civil societies. Compared to fiat money, commodity money is primitive in that its service as money seems much constrained by its physical properties such as scarcity, portability, divisibility, and recognizability. While it is folk wisdom that these properties matter, not much has been explored probably because it is not easy to place them in models many economists are used to. In other words, it may be the primitiveness of commodity money that constitutes a test to modern monetary economics.

Building on an off-shelf matching model, we offer a theory of coinage that appeals to portability and divisibility and show its usefulness to understand medieval debasements. (Our theory and modelling approach can accommodate scarcity and recognizability.) In the present model there is one sort of monetary metal but it leaves a room for another sort of metal. Two sorts of metal permit one to study issues related to bimetallism. For example, why did both Western and China choose bimetallism for centuries? Why did bimetallism in Western suddenly collapse in 1873?

[9]The absence of large denomination here will not affect our qualitative results, because such debasements only involve denominations smaller or equal to the original small denomination. We raise the small coin's silver content mainly to ease computations.

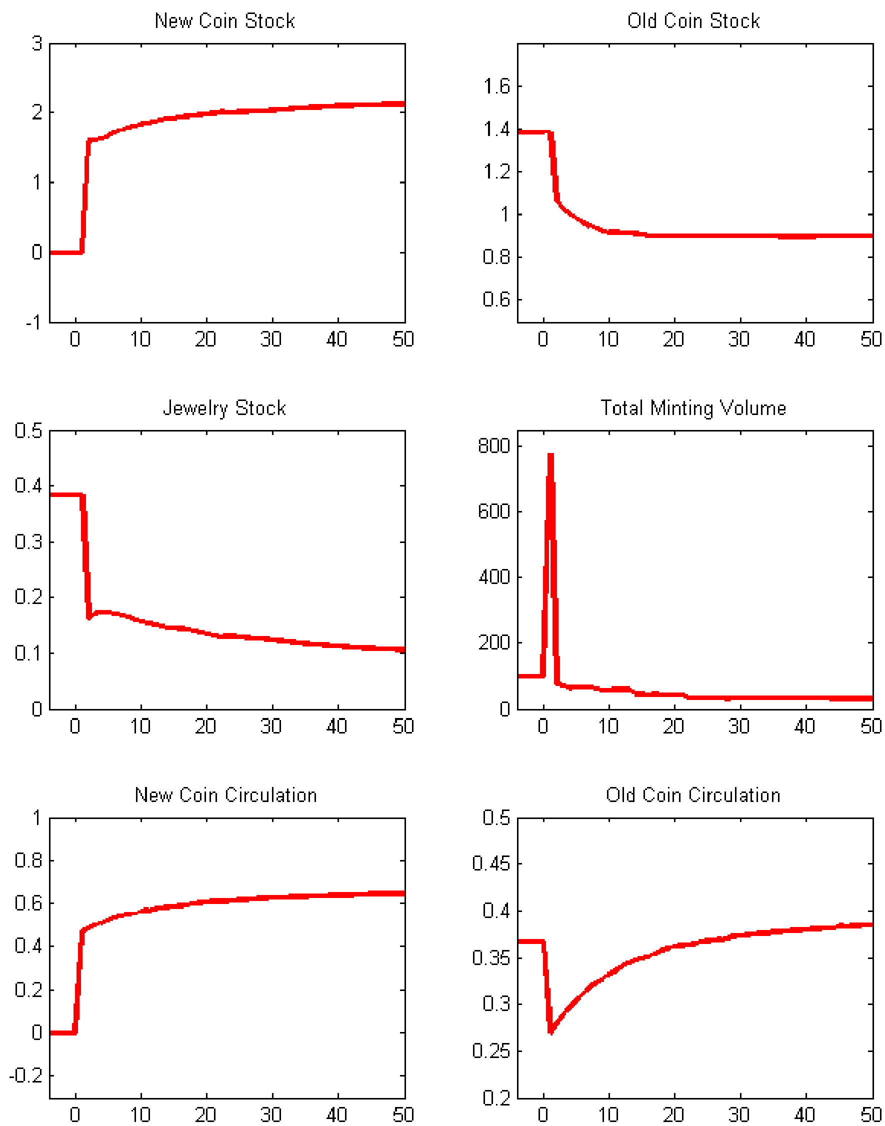

Figure 6: Transitional dynamics after the small coin is debased from containing 3 units of silver to 2.

Appendix

A. Proofs of Propositions 1 and 2

Here we give the proof of Proposition 1; the proof of Propositions 2 is the essentially the same.

Let W be the unique positive solution for z to the equation $z = 0.5u(z) + \beta z + \upsilon(B)$. Let $\mathbf{W}$ be the set of concave and nondecreasing functions from $\mathbf{Y} \times \mathbf{X}$ to $[0, W]$. Let $\mathbf{\Pi}$ and $\mathbf{\Theta}$ both be the set of measures on $\mathbf{Y} \times \mathbf{X}$ whose means are M.

For part (i), let $\mathbf{\Omega}$=$\prod_{t=0}^{\infty}[\mathbf{W}_t \times \mathbf{\Theta}_t \times \mathbf{\Pi}_{t+1}]$, where $\mathbf{W}_t \times \mathbf{\Theta}_t \times \mathbf{\Pi}_{t+1} = \mathbf{W} \times \mathbf{\Theta} \times \mathbf{\Pi}$, all t. Let the mapping $\digamma = \{\digamma_t^w, \digamma_t^\theta, \digamma_{t+1}^\pi\}_{t=0}^{\infty} : \mathbf{\Omega} \to \mathbf{\Omega}$ be defined as follows. Fix $\omega = \{w_t, \theta_t, \pi_{t+1}\}_{t=0}^{\infty} \in \mathbf{\Omega}$ and fix $t \geq 0$. Let $h_t(y, x)$ be defined by (7) and let $\Delta_1[y, x, h_t]$ and $\Delta_2[y_b, x_b, y_s, x_s, w_{t+1}]$ be the same as in the main text. Let $\digamma_t^w(\omega)(y, x) = g(y, x, h_t)$. Let $\digamma_t^\theta(\omega)$ be the set of measures defined by the right side of (8) for each $\sigma \in \Delta_1[y, x, h_t]$. Let $\digamma_{t+1}^\pi(\psi)$ be the set of measures defined by the right side of (9) for each $\delta \in \Delta_2[y_b, x_b, y_s, x_s, w_{t+1}]$. Let $\mathbf{\Omega}$ be equipped with the product topology so that it is compact. It is standard to show that $(\digamma_t^w, \digamma_t^\theta, \digamma_{t+1}^\pi)$ is upper hemi continuous, compact valued, and convex valued for each t. It follows that $\digamma$ is upper hemi continuous, compact valued, and convex valued, and, hence, that $\digamma$ has a fixed point. This fixed point is an equilibrium.

To show that this equilibrium is a monetary equilibrium, suppose by contradiction the opposite. Without loss of generality, suppose that some agent holds silver wealth B at date 0 and all his wealth is in jewelery. Consider two options of this agent: minting one unit coin 1 for certain and no minting any coin at all. For the first option, his expected payoff is bounded below by

$$\upsilon(B - m_1) - (\phi_1 + \gamma_1\} + 0.5\frac{B - M}{B}u[\frac{\beta}{1-\beta}(\upsilon(B) - \upsilon(B - m_1))]$$
$$+ 0.5\frac{\beta}{1-\beta}\upsilon(B - m_1) + 0.5\frac{\beta}{1-\beta}\upsilon(B).$$

For the second option, his expected payoff is $\upsilon(B) + \frac{\beta}{1-\beta}\upsilon(B)$. But then (11) implies the first option has a higher payoff, a contradiction. This proves part (i).

For part (ii), the proof is similar to but simpler than the proof for part (i). It is simpler because the mapping corresponding to $\digamma$ is defined on on the finite-dimensional space $\mathbf{W} \times \mathbf{\Theta} \times \mathbf{\Pi}$ instead of $\mathbf{\Omega}$. This mapping has a fixed point (w, θ, π) and the fixed point is a steady state.

B. Numerical algorithms

B1. Computing steady states of the basic model

To begin with, vectorize the $K+1$-state space into a one-dimensional state, and define the value vectors $\{w, g\}$ and distribution vectors $\{\theta, \pi\}$ accordingly. Denote the total possible number of states as S.

1. Begin with an initial guess $\{w^0, h^0, \theta^0, \pi^0\}$, where π^0 and θ^0 are consistent with the total silver stock M.

2. Given end-of-stage-1 value h^i and beginning-of-stage-1 distribution π^i from i-th iteration, solve the linear programming problem (2), and use the solution to update beginning-of-stage-1 value w^{i+1} and end-of-stage-1 distribution θ^{i+1} .

3. With w^{i+1} and θ^{i+1}, solve the pairwise bargaining problem as described in (3).[10] Record the terms of trade of each relevant pairs, and update h^{i+1} and π^{i+1} accordingly.

4. Repeat step 2-3 until the convergence criterion is satisfied: $\|w^{i+1} - w^i\| < 10^{-6}$, $\|h^{i+1} - h^i\| < 10^{-6}$ and $\|\theta^{i+1} - \theta^i\| < 10^{-8}$, $\|\pi^{i+1} - \pi^i\| < 10^{-8}$.

B2. Computing steady states of the post-debasement model

The post-debasement model is similar to the basic model, except that now we have one additional restriction in the lottery game against mint. That is, $y'_o \leq y_o$, so the newly updated distribution θ^{i+1} will always have less y_0 with each iteration. Therefore, using the algorithm described in **??** may miss the real steady state distribution. We can fix the problem by modifying step 2 as follow:

- Given end-of-stage-1 value h^i and beginning-of-stage-1 distribution π^i from i-th iteration, solve the linear programming problem (2), and use

[10]Note that although there are total S^2 possible pairs of buyer and seller, there is no need to compute that many pairs since many of the pairs have zero probability or involve a buyer with no coins.

the solution to update beginning-of-stage-1 value w^{i+1} and get a new $\widetilde{\theta}$. Then update $\theta^{i+1} = \delta \cdot \widetilde{\theta} + (1-\delta) \cdot \theta^{i}$, where $\delta \in (0,1)$.[11]

The rest of the procedure are same as A1.

B3. Computing transition paths following debasements

The computation for the transition path is essentially about iterations on the series of $\Psi \equiv \{w_t, h_t, \theta_t, \pi_{t+1}\}_{t=1}^{T}$, where T is the number of periods it takes for the economy to reach a new steady state. Before computing the transition paths, we first need to compute the post-debasement steady state using algorithm described in A2. Denote this steady state as $\{w_T, h_T, \theta_T, \pi_{T+1}\}$. We also have to translate the distribution from the pre-debasement steady state, into the beginning distribution in the debasement environment, denote the beginning distribution as π_1.

1. Take an initial guess $\Psi^0 \equiv \left\{w_t^0, h_t^0, \theta_t^0, \pi_{t+1}^0\right\}_{t=1}^{T}$, with $w_T^0 = w_T$.
2. Start from the last period T. Given w_T and θ_T^i, solve the pairwise bargaining problem as described in (3), and get h_T^i. Record the implied Markov transition matrix as Λ_T^i. Use h_T^i and π_T^i, solve the linear programming problem of minting, and get w_{T-1}^i accordingly. Record the implied Markov transition matrix as Υ_T^i. Then use w_{T-1}^i and θ_{T-1}^i, repeat the previous procedure for problems in period $T-1$. Finally, we will have a new series $\{w_t^i, h_t^i\}_{t=1}^{T}$. And then use $\{\Lambda_t^i, \Upsilon_t^i\}_{t=1}^{T}$ and π_1 and generate a new series of distributions $\left\{\pi_t^{i+1}, \theta_t^{i+1}\right\}_{t=1}^{T}$.
3. Now use $\left\{\pi_t^{i+1}, \theta_t^{i+1}\right\}_{t=1}^{T}$ and w_T, repeat Step 2 and get $\left\{\pi_t^{i+2}, \theta_t^{i+2}\right\}_{t=1}^{T}$.
4. Repeat 2-3 until the convergence criterion is met: $\max_t \left(\left\|\pi_t^{i+1} - \pi_t^i\right\|\right) < 10^{-8}$, $\max_t \left(\left\|\theta_t^{i+1} - \theta_t^i\right\|\right) < 10^{-8}$, $\max_t \left(\left\|w_t^{i+1} - w_t^i\right\|\right) < 10^{-6}$, and $\max_t \left(\left\|h_t^{i+1} - h_t^i\right\|\right) < 10^{-6}$.

C. Robustness checks

In this appendix, we show how variations in different parameters will affect the steady state of the model.

[11]In particular, we find $\delta = 0.3$ works fine.

	Total Stock			Pairwise Meeting			Circulation		Mint Vol.	
Variations	m_1	m_2	Jewelry	trading pairs	Avg pay	Avg output	m_1	m_2	m_1	m_2
Basic	3.7681	0.0619	0.3861	0.4562	1.0046	1.8509	0.4923	0.0085	0.0937	0.0797
$\gamma_k = \overline{\gamma}$	3.6851	0.0756	0.3862	0.4562	1.0042	1.8509	0.4935	0.0089	0.1018	0.0934
$\eta = 8$	3.4144	0.0379	0.5448	0.4581	1.0054	1.8736	0.4852	0.0061	0.1411	0.1475
$\phi_k = \upsilon\left(0.02\sqrt{m_k}\right)$	3.6849	0.0755	0.3862	0.4560	1.0042	1.8509	0.4935	0.0089	0.1018	0.0934
$\epsilon = 0.005$	3.8864	0.1736	0.3072	0.4628	1.0098	1.7726	0.5107	0.0109	0.1582	0.2166
$\epsilon = 0.025$	3.5003	0.0277	0.4334	0.4517	1.0022	1.9424	0.4696	0.0042	0.0962	0.0691
$\sigma = 0.4$	3.8158	0.0574	0.3840	0.4637	1.0298	1.4271	0.5073	0.0078	0.0982	0.0825
$\sigma = 0.6$	3.7358	0.1051	0.3633	0.4568	1.0000	2.3943	0.5025	0.0114	0.1087	0.1000
$\sigma_x = 0.4$	3.7646	0.0619	0.3864	0.4558	1.0037	1.8671	0.4914	0.0085	0.0936	0.0797
$\sigma_x = 0.6$	3.6889	0.0800	0.3831	0.4565	1.0049	1.8268	0.4947	0.0090	0.0997	0.0906

Table 5: Steady states with two denominations.

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