A Model of Dynamic Public Goods Contribution: Group-Size Effect*

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Abstract

How does an increase of the group size affect the public goods contribution made by group members? We analyze this question using a framework in which group members interact repeatedly and their temporary ability of producing public goods is private information. We show that the group-size effect is positive in relatively small groups, and negative in relatively large groups. This result provides a unified explanation for the previous empirical and experimental findings about the group-size effect.

1 Introduction

This paper is concerned with the dynamic aspect and cooperative behavior in the problem of private provision of public goods. In reality, many public goods problems are characterized by the repeated interaction among potential contributors. The examples include the repetition occurring in the online communities for writing product reviews, sharing movie or music files, and asking and answering questions, the computer programmer communities for developing open-source software, the problem-solving team in firms for coming up with ideas and improving production process, and the charity groups for fund-rasing. As in the nature of a

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dynamic problem, what often happens is that potential contributors can be temporarily constrained with their ability of making contributions. Members in the online community may lack useful information to share, computer programmers or workers may not have good ideas to progress the improvement, donators may be financially constrained so they cannot donate money. In this paper, we build on these two important observations and develop a framework to study the cooperation building in a contributor group. In particular, we provide an answer to one classical question in the public goods literature, which is how an increase of the group size affects cooperation, namely, the *group-size effect*.

We consider a model with infinite horizon, pure public goods and fully selfish individuals. In each period, each group member receives a personal shock on his ability of making contribution and he can make contribution only if the shock is positive. Whether he can make contribution is his private information. Most importantly, making contribution is not individually beneficial if those members only interact once. In the baseline model, we assume that one unit of contribution is enough for the whole group to consume. Group members are not allowed to communicate directly and they can only base their decisions on what has been publicly observed in the past. Since it is not of an individual member's personal interest to make contribution if the group members only interact once, the incentive for making contribution comes from the repeated interaction.

While it is not surprising that repeated interaction can provide incentive to cooperate, the mechanism through which the incentive is provided is novel here. In any period in the cooperative phase, an able member faces the following trade-off. On the one hand, he will incur a loss if he makes contribution and he can still enjoy public goods consumption if he does not but someone else does. On the other hand, it is possible that in some future period he may be unable to make contribution and his consumption will totally rely on the provision from other members. This benefit can be realized only if the cooperation in the group has not broken down by then and his free-riding behavior today may trigger the punishment and break down the cooperation. Then, under the threat of terminating cooperation, able members choose to contribute when they are sufficiently patient. Thus, we provide an explanation for why people choose to make public goods contribution even when altruism and warm glow do not play a role.

The most important result we derive is that of group-size effect. We find that, the able members' incentive to contribute becomes stronger when the group size gets larger if the initial group size is small, and the incentive becomes weaker when group size reaches a certain level and continues to expand. In short, the group-size effect is positive in the small groups but negative in the large groups. As in the standard models with pure public goods, free-riding problem becomes more severe when the group becomes large. Every able member knows that, since the group is large it is highly likely that some other member in the group is also able to make the contribution, and therefore the probability that his own deviation triggers punishment becomes smaller. The increase of group size thus has negative impact on able members' incentive. But there is also a positive impact induced by the increase of group size. Remember that the incentive to cooperate purely comes from the future benefit a member can get when he is unable but some other members are able. In a large group, the probability that some other members are able is large in any period. The later mechanism, which we call "large-scale effect" is the key force driving up able members' incentive to contribute.

Since both large-scale effect and free-riding effect are increasing when the group becomes larger, towards which direction the group-size effect goes is not immediately clear. Our result shows that the large-scale effect dominates in the small group while the free-riding effect dominates in the large group. The most intuitive way to understand this result is to consider the limit cases while ignoring the integer constraint of group size. When there is only one person in the group, he will never incur the cost as there is no future benefit to expect at all. When there is an infinite number of members in the group, no able member will contribute as the probability that his deviation triggers punishment is zero. Then, if there is a cooperation to take place, it has to happen in a medium size group.

The existing theories have diverse predictions about how group size affects public good contributions, but these predictions are usually monotone towards one direction. In the canonical setting, group members provide pure public goods and how much contribution that each individual makes depends on the amount of contribution made by other individuals (Olson, 1965, Chamberlin, 1974 and Andreoni, 1988). When the group becomes large, an individual's marginal return of supplying public goods diminishes and he will be better off by spending more resources on private consumption. That is, the free-riding problem becomes severe and the individual contribution monotonically decreases as the group size becomes larger. Although being theoretically sharp, the argument that free-riding always dominates fails to explain why individual contribution does not drop significantly when many real world communities become increasing popular. To address this inconsistency, economists recently introduce a private benefit or warm glow, such as moral satisfaction and joy of giving, into their models (Steinberg, 1987, Andreoni, 1989, 1990). Then, how group-size effect works depends on how warm glow is modeled in contributors' utility function. For example, if the warm glow increases with the number of recipients, then the free-rider problem can vanish even in a large group (Andreoni, 2007).

Empirical studies, mostly based on experimental data, gives ambiguous evidence of group-size effect. Among the recent contributions, Ledyard (1995) finds that free-riding behavior increases with the group size, Goeree, Holt and Laury (2002) find no clear group-size effect, while Zhang and Zhu (2010) find that an exogenous decrease of group size lowers the remaining contributors' average contribution on Chinese Wikipedia. The evidence of non-monotonic group-size effect is shown by Issac and Walker (1988) and Issac, Walker and Williams (1994). They investigate the free-riding problem and group-size effect in a experiment using different group size. The experiment was run for 60 decision rounds and each individual's endowment in each round was their private information. The authors find that, in the relatively small group (size from 4 to 10), the increase of group size leads to higher contribution, but in the relatively large group (size from 40 to 100), the group size effect vanishes.

We feel our result regarding group-size effect is of interest not only because it matches up well with the experiment findings shown by Issac and Walker (1988) Issac, Williams and Walker (1994), but also because it sheds light on issues on the optimal design of communities. One phenomenon we often see in the online community is that the interactions and contributions fluctuate over time and across communities. Many once prosperous communities suddenly become quiet, some of them start to revive later on and some of them collapse since then. Also, many communities start with open membership in their early stage but after becoming mature switch to restricted membership, namely only limited new members are permitted to join for a given period. Some initially free-to-join groups at some point start to charge membership fees. These observations partly suggest that the community members might tacitly use collective punishment after observing the scarcity of contribution and the community managements can restrict the group size to mitigate free-rider problem. All these observations are consistent with our theory.

Our paper is also related to other two strands of literature. First, it is related to the large literature on dynamic free-rider problems. The classical contributions include, Fershtman and Nitzan (1991) who study a dynamic public goods contribution game; Marx and Matthews (2000) who study a project-building problem in which the benefit only realizes if enough fund is raised for the project; Lockwood and Thomas (2002) who study a repeated prisoner dilemma with irreversible contribution. In all these models, the contribution today directly impacts group members' consumption tomorrow and thus are (partially) durable. Our model abstracts from this intertemporal connection and focuses on the non-durable contribution. Although the repetitive aspect is important and common in the real-world public goods provision scenarios, very little has been done on this topic. The only paper that we are aware of is Pecorino (1999) who also analyzes the group-size effect in the public goods provision using a repeated game framework. However, Pecorino assumes that every group member is able to make contribution at any time and the monitoring in the group is perfect. He focuses mainly on the large market and shows that cooperation can be sustained even when the group consists of an infinite number of individuals. One important ingredient of our model is the temporary constraint on members' ability of making contribution, which in turn gives rise to the large-scale effect. Since the ability of making contribution is members' private information, the monitoring of each group member is public but imperfect. Also, our study of group-size effect is conducted for groups with finite members which allows us to explicitly compare the incentive to contribute between two groups with different but finite sizes.

The second strand of literature to which our paper is related is the research that attempts to provide a theoretical explanation for modern firms' successful practice of using team production, especially, how the free-riding problem in teams can be mitigated. Holmstrom (1982) shows that free-riding problem is generic in the team production and a principal, who serves as a budget break, may correct such inefficiency. Different from Holmstrom (1982) and all the following up works, our model does not build on an agency framework. Kandel and Lazear (1992) develop a theory in which free-rider problem can be alleviated by the peer pressure exserted by other team members. Since all team members' payoffs depend on the total performance, members indeed have incentive to impose pressure on their team mates through physical or mental punishment if shirking is detected. Then, peer pressure may give rise to a high effort. One immediate implication that Kandel and Lazear draw is that the strategic use of peer pressure enhances effort more effectively when team members are homogeneous as they understand their co-workers' tasks and can effectively monitor each other. In contrast, our theory will work better with heterogeneous workers in the team context. Different workers coming up with different ideas at different time is the source of incentive provision in our model. An able worker will

	public good provided	public good not provided
unable member	u	0
able member	v	w

Table 1: Individual one-period payoffs.

work hard to help his team only if he knows that in some future period he has to rely on his team mates who have the idea that he does not have to enhance team performance.

The outline of the paper is as follows. Section 2 describes the setup of our baseline model. Section 3 characterizes the equilibrium. Section 4 states the main result regarding group-size effect. Section 5 extends the baseline model to various directions and examines robustness of our main finding. Section 6 concludes.

2 Model

Consider an infinite-horizon model. A group of N risk-neutral infinitely-lived individuals can produce one unit of public good in each period. Public goods are assumed not to accumulate over time. The group members cannot directly communicate or make monetary transfers.

We assume that some of the group members might be incapable to contribute to the public good production in some periods. Formally, in each period, a group member receives a personal shock which is independent and identically distributed across periods and individuals. A member is able to contribute one unit of the public good with probability q, and is not able to contribute with probability $1-q$. In what follows, we call the former an able member while the latter an unable member. N_t denotes the number of able members in period t where $N_t \in \{0, 1, ..., N\}$.

We study first a benchmark case in which individual contributions do not accumulate within one period. If at least one individual contributes in period t then the public good is produced in that period. Then an able member gets payoff $v \geq 0$ while an unable member gets $u > 0$. Additional contributions generate no extra payoffs to the group members. However, if no member contributes in period t then the public good is not provided in that period and an able member gets $w \geq 0$ while an unable member gets 0. The member's individual one-period payoffs are summarized in Table 1.

Public good provision is costly. In particular, an able member incurs a fixed cost c of providing one unit of the public good. We assume that $c > v$ which implies that the cost of production exceeds the private benefits for a contributor. It follows that no contribution will be made in a stage game. Moreover, we assume that $0 \leq w \leq v$ which means that an able member's utility from the public good consumption is weakly higher than his utility from no public good option.

We assume that the group members use symmetric pure strategies. We relax this assumption in Section 5.1 in which we consider correlated equilibrium. Suppose now that all able members contribute to the public good provision. Then the member's individual one-period *ex-ante* payoff is given by $q(v-c)+(1-q)(1-\alpha)u$, where $\alpha \equiv (1-q)^{N-1}$ is the probability that there are no other able members in the group. Suppose next that no able member contributes to the public good provision. Then the member's individual one-period *ex ante* payoff is qw. The following assumption guarantees that it is *ex ante* socially optimal that all able members contribute to the public good production.

Assumption 1 $q(v-c-w)+(1-q)(1-\alpha)u \geq 0$.

The timing of events within one period is as follows. First, able members are randomly drawn by nature. Each member's ability to contribute to the public good production is his private information. Second, able members simultaneously and independently decide whether to contribute or not. Finally, all members observe whether the public good is provided and get corresponding payoffs.

Our framework is quite general and can be used to analyze different applications. We provide now several examples and turn then to equilibrium characterization.

Peer-to-Peer Networks Consider a file-sharing network of internet users. In each period, some users (with probability q) get a rare file of value $w>0$ and choose whether to share it with the rest of the group. Sharing the file does not generate extra benefits for the users, i.e., $v=w$. The cost of sharing is c. If the file is distributed then the rest of the group gets benefit $u>0$.

Intelligence Agencies Think of a group of countries facing a common threat (e.g., terror threat). In each period, an intelligence service of each country might get a piece of relevant information which can be then revealed at cost c to intelligence services of other countries. This would obviously benefit uninformed countries ($u>0$) but would bring no extra benefit to informed ones ($v=w>0$).

Online Reviews In each period, a new restaurant is opened. Some consumers might learn by chance how good it is (with probability q). They then decide whether to post an online review about the restaurant quality. Posting a review is costly and implies no extra benefits for the informed consumers, i.e., $v=w$. Still, posting a review would benefit the uninformed consumers by generating net payoff $u > 0$ for them.

Open Source Software Consider an open source software application. With probability q each programmer comes up with an idea of how to improve the application (e.g., fixing bugs, improving efficiency, etc.) and then decides whether to make this improvement. The cost of improving the application is c. Each programmer gets benefit $v = u > 0$ if the software is upgraded, and $w = 0$ otherwise.

Teamwork Think of a team which is assigned a new task every period. With probability q each member knows how to perform the task at cost c. If the team succeeds in solving the task then each team member gets benefit $v = u > 0$. However, if the team fails then each member gets $w = 0$.

3 Equilibrium Characterization

Consider first a single stage game. In this case, able members will not contribute to the public good production since the cost exceeds their private benefits from contributing. This is the unique equilibrium in the static setting.

We turn next to the repeated setting in which all group members observe whether the public goods have been provided or not in the previous periods. The equilibrium strategy we consider is similar to that in Green and Porter (1984) and includes a cooperation phase and a punishment phase. An able member cooperates (i.e., contributes to the public good provision) in the cooperation phase and punishes (i.e., does not contribute) in the punishment phase. Punishment lasts for T periods and afterwards cooperation is restored.

We denote the state in period t by $w_t \in \{0, 1\}$. $w_t = 0$ if in period t the game is in the punishment phase while $w_t = 1$ if it is in the cooperation phase. At the end of each period, the group members observe whether a public good has been provided in this period or not, which serves as a public signal for them. We then define the following binary signal space $\left\{\underline{y}, \overline{y}\right\}$. $y_t = \overline{y}$ either if $w_{t-1} = 1$ and a period-t public good is provided, or if in period t a punishment phase has just ended. $y_t = \underline{y}$ either if $w_{t-1} = 1$ but a period-t public good is not provided, or if in period t a punishment phase has not been yet ended. The solution concept is symmetric Public Perfect Equilibrium (SPPE) in which the members condition their actions only on the public history.[1] Then all able members contribute to the public good production if in the

[1] It is well known from the literature on the repeated games with imperfect public monitoring that using public strategy is a best response to all other members' using public strategies.

previous period the public signal is $\overline{y}$, and don't contribute if in the previous period the public signal is $\underline{y}$. The transition between the states can be summarized as follows:

- $w_t = 1$ for $t = 0$;
- if $w_{t-1} = 1$ and $y_t = \overline{y}$ then $w_t = 1$; if $w_{t-1} = 1$ and $y_t = \underline{y}$ then $w_t = 0$;
- if $w_{t-1} = 0$ and $y_t = \overline{y}$ then $w_t = 1$; if $w_{t-1} = 0$ and $y_t = \underline{y}$ then $w_t = 0$.

We analyze now the able members' incentives to follow the prescribed strategy. If an able member contributes to the public good production in a punishment phase then the subsequent game will not be affected, but he gets negative net payoff $v - c - w$ in the current period. It follows that an able member has no incentive to contribute in a punishment phase and therefore will follow the prescribed strategy.

If an able member contributes to the public good production in a cooperation phase then his expected payoff is

$$v - c + \delta V^{+},$$

where $\delta \in (0, 1)$ is a common discount factor and V^{+} is the member's value function defined at the beginning of any period in the cooperation phase and before each member learns whether he is able to contribute or not. By following the prescribed strategy in the cooperation phase, the able member also ensures cooperation in the next period.

Suppose that the able member chooses to deviate in the cooperation phase. Then punishment will be trigged only if there are no other able members in the group. Therefore, the expected payoff from deviating in the cooperation phase is given by

$$(1 - \alpha) \left(v + \delta V^{+}\right) + \alpha \left(w + \delta V^{-}\right),$$

where V^{-} is the member's value function defined at the beginning of a punishment phase and before each member learns whether he is able to contribute or not. As before, $\alpha \equiv (1 - q)^{N-1}$ denotes the probability that there are no other able members in the group.

We can define the value functions V^{+} and V^{-} recursively. At the beginning of any period in a cooperation phase, a member anticipates that with probability q he will be able to contribute to the public good provision while with probability $1-q$ he will not be able to do so. Following the prescribed strategy, an able member contributes to the public good production and gets payoff $v - c + \delta V^+$. In turn, an unable member's payoff is determined by the rest of the group. If other group members are able to contribute then they follow the prescribed strategy and do contribute. This generates benefit u for the unable member and ensures cooperation in the next period. However, if there are no other able members in the group then the public good is not provided which triggers punishment. The unable member's expected payoff is δV^- in this case. It follows then that

$$V^+ = q\left(v - c + \delta V^+\right) + (1-q)\left[(1-\alpha)\left(u + \delta V^+\right) + \alpha\delta V^-\right]. \tag{1}$$

At the beginning of a punishment phase, all members realize that for T periods no public goods will be produced but afterwards cooperation will be restored. They therefore expect per-period payoff qw for T periods and $\delta^T V^+$ afterwards. It implies that

$$V^- = \sum_{\tau=0}^{T-1} \delta^\tau qw + \delta^T V^+ = \frac{1-\delta^T}{1-\delta} qw + \delta^T V^+. \tag{2}$$

Substituting (2) into (1) and rearranging yields the continuation value of cooperation V^+:

$$V^+ = \frac{(1-q)(1-\alpha)u + q(v-c) + (1-q)\alpha\delta\left(\frac{1-\delta^T}{1-\delta}\right)qw}{1 - q\delta - \delta(1-q)\left(1-\alpha+\alpha\delta^T\right)}. \tag{3}$$

An able member follows the prescribed strategy in the cooperation phase if and only if his payoff from cooperating exceeds that from deviating:

$$v - c + \delta V^+ \geq (1-\alpha)\left(v + \delta V^+\right) + \alpha\left(w + \delta V^-\right). \tag{4}$$

Substituting V^+ and V^- into (4) and simplifying yields the necessary and sufficient condition for sustaining cooperation:

$$\alpha\left(\frac{(1-\alpha)\left[(1-q)u + q(v-w)\right]}{c - \alpha(v-w)} - 1\right) \geq \frac{1-\delta}{\delta(1-\delta^T)}. \tag{5}$$

The right-hand side of (5) is a decreasing function of T. It goes to infinity when T approaches 0, which implies that without punishment, cooperation cannot be sustained. It converges to $\frac{1-\delta}{\delta}$ when a grim-trigger strategy (under which T goes to infinity) is applied. Therefore, a longer punishment phase makes cooperation easier to sustain, but it also leads to the value loss on the equilibrium path because punishment can be trigged even if no one has deviated. Since the right-hand side of (5) is always positive then a necessary condition for sustaining cooperation for given T is

$$(1-\alpha)(1-q)u + [(1-\alpha)q + \alpha](v-w) > c. \tag{6}$$

Note moreover that the right-hand side of (5) is a decreasing function of both δ and T. For any positive but finite T, it reaches its maximum of ∞ when δ approaches 0 and its minimum of $\frac{1}{T}$ when δ approaches 1. Thus for given T, as long as

$$\alpha\left(\frac{(1-\alpha)[(1-q)u + q(v-w)]}{c-\alpha(v-w)} - 1\right) \geq \frac{1}{T}, \tag{7}$$

and the members are patient enough, cooperation can be sustained. When a grim-trigger strategy is used ($T = \infty$) so that the right-hand side of (7) becomes zero, cooperation essentially requires the left-hand side of (7) to be positive. The result is summarized in the following proposition. (Proofs of this and other propositions are given in the Appendix.)

Proposition 1 *As long as condition (6) holds, there exists a threshold discount factor* $\overline{\delta} \in (0,1)$ *associated with some punishment length* $T < \infty$ *such that cooperation is sustained for all* $\delta \geq \overline{\delta}$.

Comparing (6) with Assumption 1 makes it clear that for some values of production cost c, cooperation is *ex ante* socially optimal but it can not be sustained in equilibrium.[2] Note also that applying a grim-trigger strategy ($T = \infty$) is often suboptimal as it forgoes potential surplus from cooperation by punishing forever after one period of non-provision. As long as the able members' incentives are compatible with cooperation, the shorter the punishment phase the better. Therefore, the optimal punishment length, T_δ^*, is characterized by

$$\alpha\left(\frac{(1-\alpha)[(1-q)u + q(v-w)]}{c-\alpha(v-w)} - 1\right) = \frac{1-\delta}{\delta\left(1-\delta^{T_\delta^*}\right)} \tag{8}$$

if the solution exists. Otherwise, without loss of generality, we set $T_\delta^* = \infty$.

4 Group-Size Effect

We turn next to the group-size effect to study the impacts of a group-size change on the members' incentives to cooperate. Consider first the non-deviation condition

[2] $c \in \left[(1-\alpha)(1-q)u + [(1-\alpha)q+\alpha](v-w), v-w+\frac{1-q}{q}(1-\alpha)u\right]$.

(4). Substituting (2) into (4) and rearranging yields

$$\underbrace{\alpha}_{\text{free riding}} \left[\delta \left(1 - \delta^T\right) \underbrace{V^+}_{\text{large scale}} + \left(v - w - \delta q w \frac{1 - \delta^T}{1 - \delta} \right) \right] \geq c. \tag{9}$$

The group size, N, only affects α and V^+ which are both on the left-hand side of (9). α denotes the probability that there are no other able members in the group and so decreases with N. This effect makes the left-hand side of (9) smaller as N increases. V^+ is the continuation value of cooperation and is also affected by a change in the group size N. The following lemma shows that an increase in N actually makes V^+ (and therefore the left-hand side of (9)) larger.

Lemma 1 *V^+ increases with N.*

It follows therefore that there are two opposite forces at work when the group size increases. The first force is the conventional free-riding effect reflected by α in (9). Intuitively, an able member has more incentives to deviate in a larger group. He realizes that the larger the group, the more likely there are other able members in the group and so the less likely his own deviation is to trigger punishment. The second force is what we call a large-scale effect, reflected by V^+ in (9). An able member wants cooperation to be sustained in order to enjoy public good benefits even in the periods when he will be unable to contribute and so will depend on the other members' contributions. The more members there are in the group, the more likely there will be able members in the periods when he will be unable to contribute and so the larger the continuation value of cooperation.

We use the optimal punishment length, T_δ^*, to measure the group size effect. Suppose that a finite solution to (8) exists when the group size is N'. Now increase the group size to $N'' > N'$. If the optimal punishment length T_δ^* becomes larger or even infinite under N'' then we say that the group size effect is negative. If the optimal punishment length becomes shorter after the group size increases to N'', the group-size effect is positive.

Consider the left-hand side of equation (8). If $c \geq q(v - w) + (1 - q)u$ then it is nonpositive for all $N \geq 2$ and thus cooperation can not be sustained for any group size. If $c < q(v - w) + (1 - q)u$ then the left-hand side of (8) is positive for some $N \geq 2$. We show that it has an inverted-U shape (i.e., first increases but then decreases in N) if $c > \widehat{c}$ where

$$\widehat{c} \equiv \frac{1}{2}\left(v - w - (1-q)(1-2q)(u - v + w)\right) + \frac{1}{2}\sqrt{\left((1-q)u + q(v-w)\right)\left(u - (u-v+w)\left(5q - 8q^2 + 4q^3\right)\right)}.$$

However, if $c \leq \widehat{c}$ then the left-hand side of (8) strictly decreases in N. The results are summarized in the following proposition.

Proposition 2 *If $\widehat{c} < c < q(v-w) + (1-q)u$ then the group-size effect is positive in small groups but negative in large groups. If $c \leq \widehat{c}$ then the group-size effect is always negative.*

According to Proposition 2, for $c \leq \widehat{c}$, free-riding effect prevails regardless of the group size. However, for $\widehat{c} < c < q(v-w) + (1-q)u$, the relationship between the group size and cooperative incentive is non-monotonic. While an increase in the group size intensifies both the free-riding incentives and the continuation value of cooperation, its aggregate impact depends on the current group size. We show that an increase in the group size enhances cooperation in relatively small groups but hinders cooperation in relatively large groups. Indeed, in a small group, an individual deviation is quite likely to trigger punishment and the continuation value of cooperation is low. As the group size increases, an individual deviation is somewhat less likely to trigger punishment while the continuation value of cooperation increases considerably. An increase in the group size then boosts the value of cooperation more than it boosts the free-riding incentives. In contrast, in large groups, a group-size increase just slightly affects the continuation value of cooperation and so enhances the free-riding incentives more than cooperative incentives. It follows therefore that large-scale effect dominates free-riding effect only when the group size is relatively small.

Proposition 2 therefore suggests that in case of $\widehat{c} < c < q(v-w) + (1-q)u$, for any patience level, cooperation is easier to be sustained in medium-size groups in which free-riding incentives and cooperative incentive are well balanced. In addition to a sound theoretical contribution, this result provides a rationale for restrictive membership or high membership fees which some real-world communities use in order to limit the group size. Indeed, the trade-off between large-scale effect and free-riding effect implies that as a group size grows, an increase in the probability of having more able members comes at a cost of lower individual incentives, which does not pay off in relatively large groups.

5 Discussion and Extensions

In this section, we relax some of the important assumptions of the model and discuss robustness of our results. We first consider the case in which the group members condition their contribution decisions on their observation of a signal. This allows them to somewhat coordinate their decisions and so makes cooperation easier to be sustained. We next relax the assumption of non-cumulative individual contributions and study the case of linear public good technology. We show that under some mild assumptions, our result about group-size effect holds in these extensions of the baseline model.

5.1 Correlated Equilibrium

In the baseline model, we assume that the group members cannot directly communicate with each other. Though realistic in some situations, this assumption might be too restrictive in some others. Moreover, one might underestimate the level of cooperation in a group if coordination is completely ruled out. Indeed, in our baseline model, every able member contributes along the equilibrium path while only one contribution is needed for public good provision. Therefore, all but one contributions are wasted which substantially reduces the value of cooperation.

In this section, we keep the assumption of no direct communication but suppose that the group members base their contribution decisions on their observation of a signal. Assume that the nature randomly determines an order of contribution making. For example, member 1 is assigned to make contribution first, member 2 is second, ..., member i is ith. Denote by $\mathcal{O}$ the finite set of all possible orders. We assume that all orders in $\mathcal{O}$ are realized with the equal probability. In each period, a member observes the signal only when it is his turn to make contribution and no contribution has been made so far in that period. The period ends either when one contribution has been made or when all group members have been called for contribution making. At the end of the period, all members know whether the contribution has been made or not. The prescribed equilibrium strategy requires that an able member makes contribution if he observes the signal. The punishment will be triggered if no contribution is made in the previous period. We characterize correlated equilibrium in which no group member wants to deviate from the prescribed strategy if the others don't deviate.[3]

[3] Alternatively, one could consider mixed-strategy equilibrium in which able members randomize between making and not making contribution.

Although the group members still cannot directly communicate with each other, the signal mechanism works as a coordinating device and guarantees that at most one contribution per period is made along the equilibrium path. Therefore, the contribution waste of the baseline model does not arise here, which implies higher value of cooperation.

Think of the following interpretation of this setting. If each period, there might be several members able to contribute to the public good provision. Since the probability of two members making contribution at exactly the same time is tiny, it is plausible to assume that the timing of contribution making is sequential. The explanation for this might be different time availability assigned randomly to the group members. Then an able member contributes to the public good production only if he observes that no contribution is made so far by his peers. However, if at the moment of contributing an able member observes that one contribution has been already made then he does not need to contribute again. For example, if an informed member reveals a piece of useful information to the rest of the group then other initially informed members will not do so.

We turn next to the analysis of the able members' incentives to follow the prescribed strategy. (Indeed, an unable member cannot make contribution even if he observes the signal.) Similarly to the baseline model studied in Section 3, an able member here has no incentive to contribute in a punishment phase and so will follow the prescribed strategy. Consider now his incentives in a cooperation phase. Suppose that an able member receives the signal and so knows that it is his turn to contribute. The expressions for the expected payoffs from cooperating and deviating are the same as in the baseline model except for the value functions which we denote here by $\tilde{V}^+$ and $\tilde{V}^-$. Obviously, the value function defined at the beginning of a punishment phase satisfies $\tilde{V}^- = V^-$. The value function defined at the beginning of any period in a cooperation phase, $\tilde{V}^+$, is

$$\tilde{V}^+ = q\left(v - \beta c + \delta V^+\right) + (1-q)\left[(1-\alpha)\left(u + \delta V^+\right) + \alpha\delta V^-\right],$$

where β is the probability that conditional of being able, a member is the first one among all able members to receive the signal. β is given by

$$\beta \equiv \sum_{k=0}^{N-1} \binom{N-1}{k} q^k (1-q)^{N-1-k} \frac{1}{k+1} = \frac{1-(1-q)\,\alpha}{Nq}.$$

Note that along the equilibrium path in our correlated equilibrium, only the able member who receives the signal first, actually incurs the cost c of public good production.

Following the similar steps as in Section 3 yields the non-deviation condition for an able member who receives the signal:

$$\alpha\left(\frac{(1-\alpha)\left[(1-q)\,u+q\,(v-w)\right]+(1-\beta)\,cq}{c-\alpha\,(v-w)}-1\right)\geq\frac{1-\delta}{\delta(1-\delta^{T})}. \tag{10}$$

It is natural to expect that in correlated equilibrium, cooperation is easier to be sustained than in the baseline model. Indeed, the signal mechanism generates at most one contribution per period and so enhances the value of cooperation. As a result, the left-hand side of (10) is strictly greater than that of (5).

We turn now to our main research question, namely, the group-size effect. It is *a priori* not clear how an increase in the group size affects the members' incentives to cooperate. To answer this question, we study the impacts of a change in N on the left-hand side of non-deviation condition (10). Note that on the left-hand side of (10), both α and β depend on N, and α enters both in the numerator and denominator, which considerably complicates the analysis of the general case. Then, to obtain a clear-cut result, we restrict our attention to the case in which an able member gets the same level of utility from the public good consumption and from no public good option, $v=w$ (as in the examples of information sharing).

Consider the left-hand side of (10) for $v=w$. If $c\geq u$ then it is nonpositive for all $N\geq 2$ and therefore cooperation can not be sustained for any group size. If $c<u$ then it is positive for some $N\geq 2$. We show that it has an inverted-U shape (i.e., first increases and then decreases in $N\geq 2$) either if $q\leq\frac{1}{2}$ or if $q>\frac{1}{2}$ and $c>\widehat{f}\,(q)\,u$ where

$$\widehat{f}\,(q)\equiv\frac{4\,(1-q)\,(1-2q)\ln\left(1-q\right)}{(2-q)\,q-2\,(1+2q\,(1-q))\ln\left(1-q\right)}.$$

However, if $q>\frac{1}{2}$ and $c\leq\widehat{f}\,(q)\,u$ then the left-hand side of (10) strictly decreases in $N\geq 2$. We formalize the results in the following proposition.

Proposition 3 *If $q\leq\frac{1}{2}$ and $0<c<u$, or if $q>\frac{1}{2}$ and $\widehat{f}\,(q)\,u<c<u$, then the group-size effect is positive in small groups and negative in large groups. If $q>\frac{1}{2}$ and $0<c\leq\widehat{f}\,(q)\,u$ then the group-size effect is always negative.*

Proposition 3 suggests that for a relatively large range of parameter values, the main insights of our baseline model also hold in the case of correlated equilibrium.

Here, because of implicit coordination generated by the signal mechanism, the probability of an able member contributing is strictly lower than 1. The more members there are in the group, the less likely an able member is to receive the signal to contribute. Therefore, the large-scale effect is amplified here relative to the baseline case. As a result, in relatively small groups, the large-scale effect again dominates the free-riding effect. Still, the important trade-off between the two effects highlighted in the previous section remains true in the correlated equilibrium. Indeed, in relatively large groups, the free-riding effect dominates the (even amplified) large-scale effect since deviation is quite unlikely to trigger punishment and thus incentives to free ride are stronger than those to cooperate.

5.2 Cumulative Contributions

In the baseline model, we assume that individual contributions do not accumulate within one period. We relax this assumption here to consider the setting in which public good consumption strictly increases in the number of individual contributions, as in the cases of dispersed information, ideas and labor input. So, if each able member cooperates, an increase in the group size affects not only the probability of a single unit being provided but the expected total number of contributions. Our final goal is then to study the group-size effect in this environment.

Consider the following production technology. If the total number of contributions is $k = 0, 1, ..., N$ then the public good of size $P(k)$ is provided and each group member gets utility $P(k)$ from consuming it. So the public good production is deterministic. Alternatively, $P(k)$ might be interpreted as the probability of the public good being successfully provided. We assume that $P(k)$ is strictly increasing in k and that $P(0) = 0$.

In each period, an able member decides whether to contribute one unit to the public good production or not. The cost of contributing is $c > 0$. To be consistent with our baseline model, we assume that $P(k) - P(k-1) < c$ for $k = 1, ..., N$. So, an able member has no incentive to make contribution in a stage game. Then, the unique equilibrium in the static setting entails no public good contribution.

We study next the repeated setting in which all group members observe the total number of contributions made in the previous periods. As in the baseline model, the solution concept here is SPPE. To characterize SPPE, we follow the approach developed by Abreu et al. (1986, 1990). According to this approach, a SPPE value can be decomposed into a member's current payoff and the continuation value, denoted by $v(k)$, which is a mapping from the set of all public outcomes, $k = 0, 1, ..., N$, into the set of SPPE values, $V^*(N, \delta)$. Note that each able member choosing not to contribute constitutes a SPPE and so $V^*(N, \delta)$ is non-empty: $0 \in V^*(N, \delta)$. It is also assumed that the group members have access to a public randomization device at the end of each period. Then $v(k)$ can be a probability distribution over $V^*(N, \delta)$. As a consequence, $V^*(N, \delta)$ is a convex set. Then, applying Abreu et al.'s approach, we can show that $V^*(N, \delta)$ is a closed interval of the form $[0, \overline{v}(N, \delta)]$.

We focus the analysis on the upper bound of $V^*(N, \delta)$, $\overline{v}(N, \delta)$, which is the highest value a SPPE can achieve. We believe it is reasonable to do so since the SPPE with the highest value will be a natural candidate in case the group members have ever a chance to coordinate on equilibrium selection. To construct a SPPE, we also have to choose a continuation value from $V^*(N, \delta)$. Abreu et al. (1986, 1990) show that any value in $V^*(N, \delta)$ can be achieved by the following trigger strategy with the *bang-bang* property. At the beginning of the game, every able member contributes to the public good provision. Then, after observing the public outcome k, with probability $1 - \eta(k)$ all group members continue to play the same cooperative strategy in the next period, and with probability $\eta(k)$ they switch to static Nash forever. We can then find the highest value, $\overline{v}(N, \delta)$, among those trigger strategy SPPE values.

We assume without loss of generality that $P(k) = k$. The following problem then characterizes the highest SPPE value, $\overline{v}(N, \delta)$:

$$
\begin{aligned}
&\overline{v}(N, \delta) = \max_{v, \eta(k)} v \\
&\text{s.t.} \quad v = qv^a + (1-q)v^u, \\
&\qquad v^a = (N-1)q + 1 - c + \delta v \sum_{k=0}^{N-1} (1 - \eta(k+1)) \binom{N-1}{k} q^k (1-q)^{N-k-1}, \\
&\qquad v^u = (N-1)q + \delta v \sum_{k=0}^{N-1} (1 - \eta(k)) \binom{N-1}{k} q^k (1-q)^{N-k-1}, \\
&\qquad v^a \geq v^u.
\end{aligned}
$$

Here, v^a is an able member's discounted payoff if he follows the prescribed strategy while v^u is an unable member's discounted payoff. In each period in which cooperation is sustained, every member expects other members to provide $(N-1)q$ units of public good. Moreover, an able member also contributes one unit at cost c. These are the members' current payoffs. The discounted continuation value of cooperation is δv multiplied by the probability of cooperation being sustained. The *ex ante* expected value v is thus $qv^a + (1-q)\,v^u$. In the equilibrium, the incentive compatibility (IC) constraint has to hold so that an able member has no incentive to mimic an unable member: $v^a \geq v^u$.

Substituting the expressions for v^a and v^u into v and the IC constraint and rearranging yields

$$\begin{aligned} v &= q(1-c) + (N-1)\,q + \delta v \sum_{k=0}^{N} \binom{N}{k} q^k (1-q)^{N-k} (1-\eta(k)), \quad (11) \\ c-1 &\leq \delta v \sum_{k=0}^{N-1} \binom{N-1}{k} q^k (1-q)^{N-k-1} (\eta(k) - \eta(k+1)). \end{aligned}$$

In the equilibrium, the IC constraint has to hold with equality. We prove it by contradiction. For the IC constraint to be satisfied, some $\eta(k)$ have to be strictly greater than zero. Suppose that the IC constraint does not bind in the equilibrium. Then, there exists at least one positive $\eta(k)$ which can be decreased by a tiny amount $\varepsilon > 0$ while the IC constraint is still satisfied. However, decreasing this $\eta(k)$ will increase v (since v is a decreasing function of all $\eta(k)$). Thus, this was not an equilibrium, which leads to a contradiction.

In what follows, we use the method developed by Abreu et al. (1991). According to this method, solving for $\overline{v}(N,\delta)$ is equivalent to finding the maximum likelihood test to detect deviation. We next show this formally. Define

$$\mathcal{L} \equiv \frac{q \sum_{k=0}^{N-1} \binom{N-1}{k} q^k (1-q)^{N-1-k} \eta(k) + (1-q) \sum_{k=0}^{N-1} \binom{N-1}{k} q^k (1-q)^{N-1-k} \eta(k)}{q \sum_{k=0}^{N-1} \binom{N-1}{k} q^k (1-q)^{N-1-k} \eta(k+1) + (1-q) \sum_{k=0}^{N-1} \binom{N-1}{k} q^k (1-q)^{N-1-k} \eta(k)}.$$

The numerator is the *ex ante* probability of triggering punishment if one able member deviates and does not make contribution. This probability is then the measure of punishment off the equilibrium path. The denominator is the *ex ante* probability of triggering punishment if no member deviates. This probability is therefore the measure of punishment along the equilibrium path and represents the equilibrium value loss. Then the likelihood ratio $\mathcal{L}$ measures how effective the punishment is in deterring deviation per unit of the value loss along the equilibrium path. The larger $\mathcal{L}$ is, the heavier punishment is imposed on the deviator per unit of the equilibrium value loss.

Since

$$\mathcal{L}-1=\frac{q\sum\limits_{k=0}^{N-1}\binom{N-1}{k}q^{k}(1-q)^{N-1-k}(\eta(k)-\eta(k+1))}{\sum\limits_{k=0}^{N}\binom{N}{k}q^{k}(1-q)^{N-k}\eta(k)}, \tag{12}$$

then the binding IC constraint can be written as a function of $\mathcal{L}-1$ in the following way:

$$c-1=\frac{\delta v}{q}\left(\mathcal{L}-1\right)\sum_{k=0}^{N}\binom{N}{k}q^{k}\left(1-q\right)^{N-k}\eta\left(k\right).$$

Clearly, $\mathcal{L}$ has to be strictly greater than 1 for the IC constraint to hold. Substituting the IC constraint into the value function (11) yields

$$v=\frac{1}{1-\delta}\left[q\left(N-1\right)-q\left(c-1\right)\left(1+\frac{1}{\mathcal{L}-1}\right)\right], \tag{13}$$

which after rearranging becomes

$$v=\frac{1}{1-\delta}\left[\left(1-q\right)q\left(N-1\right)+q\left(q\left(N-1\right)+1-c\right)\right]-\frac{1}{1-\delta}\frac{q\left(c-1\right)}{\mathcal{L}-1}.$$

The first term above is the expected value in case cooperation can be sustained forever. The second term is the expected loss resulting from future punishments along the equilibrium path. (13) implies that v is a strictly increasing function of $\mathcal{L}$. Therefore, the optimal punishment scheme maximizes $\mathcal{L}$ subject to the binding IC constraint, which is equivalent to finding the maximum likelihood test for detecting deviation.

We prove in the Appendix that $\mathcal{L}$ reaches its maximum value when $\eta(0)>0$, $\eta(1)=...=\eta(N)=0$; its second largest value when $\eta(0)=1$, $\eta(1)>0$, $\eta(2)=...=\eta(N)=0$; its third largest value when $\eta(0)=\eta(1)=1$, $\eta(2)>0$, $\eta(3)=...=\eta(N)=0$; ...; its kth largest value when $\eta(0)=...=\eta(k-2)=1$, $\eta(k-1)>0$, $\eta(k)=...=\eta(N)=0$. Then the optimal punishment scheme is characterized by the public outcome $\tilde{k}\in\{0,1,...,N\}$ such that $\eta(k)=1$ for $k<\tilde{k}$, $0<\eta\left(\tilde{k}\right)<1$, $\eta(k)=0$ for $k>\tilde{k}$. $\tilde{k}$ and $\eta\left(\tilde{k}\right)$ are found through the following procedure:

- set $\tilde{k}=0$ and check the IC constraint for $0<\eta(0)<1$, $\eta(k)=0$, $k>0$; if the IC constraint holds then this is the optimal punishment scheme and $\eta(0)$ is characterized by the binding IC constraint; if the IC constraint does not hold then

- set $\tilde{k}=1$ and check the IC constraint for $\eta(0)=1$, $0<\eta(1)<1$, $\eta(k)=0$,

$k > 1$; if the IC constraint holds then this is the optimal punishment scheme and $\eta(1)$ is characterized by the binding IC constraint; if the IC constraint does not hold then

- set $\tilde{k} = 2$ and check the IC constraint for $\eta(0) = \eta(1) = 1$, $0 < \eta(2) < 1$, $\eta(k) = 0$, $k > 2$; if the IC constraint holds then this is the optimal punishment scheme and $\eta(2)$ is characterized by the binding IC constraint; if the IC constraint does not hold then

- move to $\tilde{k} = 3$, check the IC constraint, and continue this process until the IC constraint is satisfied for some $\tilde{k}$.

The following proposition summarizes the result.

Proposition 4 *The optimal SPPE value, $\overline{v}(N, \delta)$, is supported by the following cut-off strategy: $\eta(k) = 1$ for $k < \tilde{k}$, $0 < \eta\left(\tilde{k}\right) < 1$, $\eta(k) = 0$ for $k > \tilde{k}$. $\tilde{k}$ and $\eta\left(\tilde{k}\right)$ are found through the procedure described above.*

According to Proposition 4, the optimal punishment scheme is characterized by a public outcome $\tilde{k}$ such that $\eta(k) = 1$ for $k < \tilde{k}$, $0 < \eta\left(\tilde{k}\right) < 1$, $\eta(k) = 0$ for $k > \tilde{k}$. Substituting the optimal punishment scheme into (12) yields

$$\mathcal{L} - 1 = \frac{q\left[\binom{N-1}{\tilde{k}-1} q^{\tilde{k}-1}(1-q)^{N-\tilde{k}}\left(1-\eta\left(\tilde{k}\right)\right) + \binom{N-1}{\tilde{k}} q^{\tilde{k}}(1-q)^{N-\tilde{k}-1}\eta\left(\tilde{k}\right)\right]}{\sum_{k=0}^{\tilde{k}-1}\binom{N}{k} q^{k}(1-q)^{N-k} + \binom{N}{\tilde{k}} q^{\tilde{k}}(1-q)^{N-\tilde{k}}\eta\left(\tilde{k}\right)}. \tag{14}$$

We next study the group-size effect to check robustness of the baseline model results. In this setting with cumulative contributions, we measure the group-size effect by the impact of a group-size change on the optimal cut-off strategy. The group-size effect is negative if an increase in N makes $\tilde{k}$ larger or (in case $\tilde{k}$ remains constant) makes $\eta\left(\tilde{k}\right)$ larger. Indeed, in this case, a stricter punishment is needed to sustain cooperation. The group-size effect is positive if an increase in N makes $\tilde{k}$ smaller or (in case $\tilde{k}$ remains constant) makes $\eta\left(\tilde{k}\right)$ smaller. Then, cooperation can be sustained with a more lenient punishment.

Ideally, we would like to characterize the group-size effect for the whole range of parameters N, c, q and δ. Although Proposition 4 provides an algorithm for finding the upper bound of $V^*(N, \delta)$, we have no explicit expressions for $\tilde{k}$ and $\eta\left(\tilde{k}\right)$ in terms of N and other parameters. This considerably complicates our task. In what follows, we restrict our attention to the set of equilibria with $\tilde{k} = 0$ and analyze the impact of a group-size change on the IC constraint and $\eta(0)$. Substituting $\tilde{k}=0$ into (14) yields

$$\mathcal{L}-1=\frac{q(1-q)^{N-1}\eta(0)}{(1-q)^{N}\eta(0)}=\frac{q}{1-q}.$$

Then substituting $\mathcal{L}-1$ into the value function (13) yields

$$v=\frac{1}{1-\delta}\left(q(N-1)-(c-1)\right).$$

Finally, substituting $\mathcal{L}-1$ and v into the binding IC constraint and rearranging yields

$$\eta(0)=\frac{(1-\delta)(c-1)}{\delta\left(q(N-1)-(c-1)\right)(1-q)^{N-1}}.$$

The next step is to verify that $\tilde{k}=0$ indeed characterizes the optimal punishment scheme. This is the case if $\eta(0)\in(0,1)$, which amounts to

$$c-1<\frac{q(N-1)}{\frac{1-\delta}{\delta(1-q)^{N-1}}+1},$$

i.e., c being small enough.

Without loss of generality, we consider an increase in the group size from N_1 to N_1+1. The group-size effect is positive if (i)$\tilde{k}=0$ remains to be the solution after the group size becomes N_1+1; (ii)$\eta(0)$ decreases after the group size becomes N_1+1. Intuitively, if the increase of group size reduces members' incentive to deviate and thus favors cooperation, the new optimal punishment scheme should become less strict, compared to the original one. Proposition 5 shows that the insight from the benchmark model continues to hold in this more general environment if q is not too small.

Proposition 5 *Suppose the initial group size is N_1 and the cooperation sustains with $\tilde{k}=0$. If the group size increase to N_1+1, the group size effect is positive when N_1 is relatively small and negative when N_1 is relatively large if and only if $\frac{1}{2}<q<\frac{c}{2}$.*

6 Conclusion

There is an extensive theoretical literature focused on how an increase of group size affects group members' incentive to contribute to the public goods. Despite this extensive literature, however, only a few paper considers what new implications we can draw about group-size effect if the dynamic aspect is brought into the picture. This article investigates this issue by building a model that combines the repeated interaction of group members and their temporary constraints of making contribution. In our paper, an increase of the group size simultaneously increases the temptation to free ride and the future value of maintaining cooperation. In a small group the large scale effect dominates while in a large group the free-riding effect dominates. As the consequence, the group-size effect is positive in the small group but negative in the large group.

We believe that our result provides a novel angle to interpret peoples' incentive to make public goods contribution and thus how the change of group size affects their behavior. The model and the underlying mechanism is fairly simple, which makes possible a number of extensions to be done. First, in our analysis, the probability of being able and the cost of making contributions are assumed to be identical across members. It is easy to introduce heterogeneity to those parameter, and by doing so, one can explore an optimal member formation of a group (team). We have done some preliminary analysis by introducing heterogenous probabilities of being able. We find that an optimal team formation often requires a mixture of high type members and low type members as the high type members have higher incentive to deviate. A second possible extension is to have a more detailed production process. For example, in the general case with substitutive contribution, the production can be interpreted as an information aggregating process such as in voting models. An explicit information aggregation process will allow us to analyze problems such as the optimal size of deliberating committees and may yield new insight on how efficiently information is aggregated (for a related work, see Koriyama and Szentes, 2009). Finally, in this article, we assume that each member has a positive probability of being unable, which implies that the probability that no one is able is positive. It may be arguable that in a fairly large community, the probability that no one is unable is approximately zero. Then it would be worthwhile to extend our framework to an environment in which the number of able members is always positive but the identity of able member is random.

CHECK ALL REFERENCES, YEARS, ETC

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Appendix

Proof of Proposition 1

Note that the right-hand side of (5) is a continuous function, strictly decreasing in both δ and T. If $T = \infty$ then the right-hand side of (5) becomes $\frac{1-\delta}{\delta}$, which strictly decreases from ∞ to 0 when δ increases from 0 to 1. Then, from the Intermediate Value Theorem, there exists a $\overline{\delta} \in (0, 1)$ such that

$$\alpha \left(\frac{(1-\alpha)\left[(1-q)\, u + q\,(v-w)\right]}{c - \alpha\,(v-w)} - 1 \right) = \frac{1-\overline{\delta}}{\overline{\delta}}.$$

For any $\delta > \overline{\delta}$, we have

$$\frac{1-\delta}{\delta} < \frac{1-\overline{\delta}}{\overline{\delta}}.$$

Since the right-hand side of (5) is continuous in T, there always exists a $T < \infty$ such that

$$\alpha \left(\frac{(1-\alpha)\left[(1-q)\, u + q\,(v-w)\right]}{c - \alpha\,(v-w)} - 1 \right) \geq \frac{1-\delta}{\delta\left(1-\delta^{T}\right)}.$$

The proposition then follows. *Q.E.D.*

Proof of Lemma 1

Differentiating (3) with respect to α and simplifying yields

$$\operatorname{sign}\left(\frac{dV^{+}}{d\alpha}\right) = -\operatorname{sign}\left(q\delta\left(1-\delta^{T}\right)(-c-u-w+v) + \left(1-\delta^{T+1}\right)u\right).$$

Consider the argument of the sign function, $q\delta\left(1-\delta^{T}\right)(-c-u-w+v) + \left(1-\delta^{T+1}\right)u$, as a function of T. It is easy to check that its first-order derivative is proportional to $\delta^{T+1}\ln\delta$, which implies that the argument itself is monotonic in T. When $T = 0$, it equals to $(1-\delta)\,u \geq 0$. When $T = \infty$, it equals to $q\delta(-c-u-w+v)+u$ which is strictly positive because

$$\begin{aligned} q\delta(-c-u-w+v)+u &= q\delta\,(v-w-c) + (1-q\delta)\,u \\ &\geq q\delta\,(v-w-c) + (1-\alpha)(1-q\delta)\,u > 0. \end{aligned}$$

The last inequality follows from Assumption 1. Therefore, the argument of the sign function is always positive for $T \in [0, \infty]$ and

$$\frac{dV^+}{d\alpha} < 0.$$

Since α is a decreasing function of N then

$$\frac{dV^+}{dN} > 0.$$

Q.E.D.

Proof of Proposition 2

We use the first-order condition with respect to α to find critical points of the left-hand side of (5). There are two critical points:

$$\alpha_1 = \frac{1}{v-w}\left[c - \frac{\sqrt{c(1-q)(c-v+w)(u-v+w)((1-q)u+q(v-w))}}{(1-q)(u-v+w)}\right]$$

and

$$\alpha_2 = \frac{1}{v-w}\left[c + \frac{\sqrt{c(1-q)(c-v+w)(u-v+w)((1-q)u+q(v-w))}}{(1-q)(u-v+w)}\right].$$

α_2 is strictly larger than 1 and is therefore ruled out since $\alpha \in [0, 1]$. α_1 is always smaller than 1. The second-order condition with respect to α is

$$\frac{-2c(c-v+w)[(1-q)u+q(v-w)]}{(c-\alpha(v-w))^3} < 0.$$

Therefore, the left-hand side of (5) is a strictly concave function of $\alpha \in [0, 1]$. α_1 is smaller than or equal to 0 if and only if $q(v-w)+(1-q)u \leq c$. In this case, the left-hand side of (5) is nonpositive for $\alpha \in [0, 1]$ and therefore cooperation can not be sustained for any group size N. α_1 is larger than 0 if and only if $q(v-w)+(1-q)u > c$. Then the left-hand side of (5) has a single peak at $(0, 1)$ and is therefore first increasing and then decreasing in α. Since α is a strictly decreasing function of N then the left-hand side of (5) is also first increasing and then decreasing in N.

However, strictly speaking, α cannot exceed $1-q$ since at least two members are required to form a group. Thus, in order to prove that the left-hand side of (5) has an inverted-U shape one has to show that $\alpha_1 \in (0, 1-q)$. Then condition

$$2c > v - w - (1-q)(1-2q)(u-v+w) + \sqrt{((1-q)u + q(v-w))(u-(u-v+w)(5q-8q^2+4q^3))}$$

ensures that $\alpha_1 < 1-q$. However, if the above condition does not hold then $\alpha_1 \geq 1-q$ and thus the left-hand side of (5) is an increasing function of $\alpha \in (0, 1-q)$ and a decreasing function of $N \geq 2$. *Q.E.D.*

Proof of Proposition 3

Substituting $v = w$, α and β into the non-deviation condition (10) yields

$$(1-q)^{N-1} \frac{Nu(1-q)\left(1-(1-q)^{N-1}\right) - c\left(1-(1-q)^N + N(1-q)\right)}{cN} \geq \frac{1-\delta}{\delta(1-\delta^T)}. \tag{15}$$

Consider the left-hand side of (15). Its first-order derivative is given by

$$\frac{(1-q)^{N-1}\left(c\left(1-(1-q)^N + N\ln(1-q)\left[-N(1-q)+2(1-q)^N-1\right]\right) + uN^2\ln(1-q)\left[1-q-2(1-q)^N\right]\right)}{cN^2}. \tag{16}$$

It is easy to check that the sign of (16) is equal to the sign of

$$\frac{c}{u} + \widehat{f}(N,q),$$

where

$$\widehat{f}(N,q) \equiv \frac{N^2\ln(1-q)\left[1-q-2(1-q)^N\right]}{1-(1-q)^N + N\ln(1-q)\left[-N(1-q)+2(1-q)^N-1\right]}.$$

We next compare $\frac{c}{u}$ with $-\widehat{f}(N,q)$ to determine the sign of (16). For any $q \in (0,1)$, $-\widehat{f}(N,q)$ is an increasing function of $N \geq 2$. It takes its minimum of $\widehat{f}(q) \equiv \widehat{f}(2,q) = \frac{4(1-q)(1-2q)\ln(1-q)}{(2-q)q-2(1+2q(1-q))\ln(1-q)}$ when $N = 2$ and approaches 1 from below when N goes to infinity. Consider the following three cases.

1. $\frac{c}{u} \geq 1$.

   In this case, $\frac{c}{u} + \widehat{f}(N,q) > 0$ and so the sign of (16) is strictly positive for all $N \geq 2$. So, the left-hand side of (15) is an increasing function of $N \geq 2$ bounded between $-\frac{1}{2}(1-q)\left(2-q^2-2(1-q)q\frac{u}{c}\right) < 0$ and 0, and so nonpositive for all $N \geq 2$. Thus, cooperation can not be sustained for any group size.

2. $\max\left[\widehat{f}(q), 0\right] < \frac{c}{u} < 1$, which amounts to $q \leq \frac{1}{2}$ and $c < u$, or $q > \frac{1}{2}$ and $\widehat{f}(q)\,u < c < u$.

   In this case, there exists a unique $\overline{N} > 2$ such that $\frac{c}{u} + \widehat{f}\left(\overline{N}, q\right) = 0$ and $\frac{c}{u} + \widehat{f}(N, q) \gtreqless 0$ for $N \lesseqgtr \overline{N}$. Thus, the left-hand side of (15) has a single peak at $\overline{N} > 2$ and approaches zero from above when N goes to infinity. It has therefore an inverted-U shape at $N \geq 2$.

3. $0 < \frac{c}{u} \leq \max\left[\widehat{f}(q), 0\right]$, which amounts to $q > \frac{1}{2}$ and $c \leq \widehat{f}(q)\,u$.

   In this case, $\frac{c}{u} + \widehat{f}(N, q) < 0$ and so the sign of (16) is strictly negative for all $N \geq 2$. The left-hand side of (15) is a decreasing function of $N \geq 2$ bounded between $-\frac{1}{2}(1-q)\left(2 - q^2 - 2(1-q)\,q\frac{u}{c}\right) > 0$ and 0, and so positive for all $N \geq 2$.

To sum up, the left-hand side of (15) has an inverted-U shape at $N \geq 2$ either when $q \leq \frac{1}{2}$ and $c < u$ or when $q > \frac{1}{2}$ and $\widehat{f}(q)\,u < c < u$. It decreases in $N \geq 2$ when $q > \frac{1}{2}$ and $c \leq \widehat{f}(q)\,u$. *Q.E.D.*

Proof of Proposition 4

Some $\eta(k)$ have to be positive for the IC constraint to hold. Otherwise, v^u will be strictly higher than v^a and therefore the IC constraint will be violated. Since some $\eta(k)$ are positive, $\mathcal{L}$ is well defined.

To find the highest $\mathcal{L}$, we need to find a public outcome k which is most likely to occur off the equilibrium path relative to its likelihood on the equilibrium path. For each public outcome, the power of test, $\mathcal{L}(k)$, is defined as the likelihood ratio of triggering punishment after public outcome k has been observed. Denote by $\theta^d(k)$ the probability of triggering punishment after observing k off the equilibrium path and by $\theta^c(k)$ the probability of triggering punishment after observing k on the equilibrium path. Then

$$\mathcal{L}(k) = \frac{\theta^d(k)}{\theta^c(k)} = \frac{q\cdot\begin{pmatrix}N-1\\k\end{pmatrix} q^k(1-q)^{N-1-k} + (1-q)\cdot\begin{pmatrix}N-1\\k\end{pmatrix} q^k(1-q)^{N-1-k}}{q\cdot\begin{pmatrix}N-1\\k-1\end{pmatrix} q^{k-1}(1-q)^{N-k} + (1-q)\cdot\begin{pmatrix}N-1\\k\end{pmatrix} q^k(1-q)^{N-1-k}} = \frac{N-k}{N}\frac{1}{1-q}.$$

It is easy to check that $\{\mathcal{L}(k)\}_{k=0}^{N}$ is a sequence strictly decreasing in k. Note that

$$\mathcal{L} = \frac{\sum\limits_{k=0}^{N} \theta^d(k)\,\eta(k)}{\sum\limits_{k=0}^{N} \theta^c(k)\,\eta(k)}.$$

Moreover,

$$\frac{\sum\limits_{j=0}^{k-1} \theta^d(j) + \theta^d(k)\,\eta(k)}{\sum\limits_{j=0}^{k-1} \theta^c(j) + \theta^c(k)\,\eta(k)} > \frac{\sum\limits_{j=0}^{k} \theta^d(j) + \theta^d(k+1)\,\eta(k+1)}{\sum\limits_{j=0}^{k} \theta^c(j) + \theta^c(k+1)\,\eta(k+1)} \tag{17}$$

for $k = 0, ..., N-1$. To see this formally, compare the left-hand side of the above inequality with the right-hand side. This is equivalent to comparing

$$\begin{aligned}
&(1-\eta(k)) \sum_{j=0}^{k-1} \left(\theta^c(k)\,\theta^d(j) - \theta^d(k)\,\theta^c(j)\right) + \\
&\eta(k+1) \sum_{j=0}^{k-1} \left(\theta^c(k+1)\,\theta^d(j) - \theta^d(k+1)\,\theta^c(j)\right) + \\
&\eta(k)\,\eta(k+1)\left[\theta^d(k)\,\theta^c(k+1) - \theta^c(k)\,\theta^d(k+1)\right]
\end{aligned} \tag{18}$$

with zero. The first term equals zero for $k = 0$ and is strictly positive for $k = 1, ..., N-1$ since $\frac{\theta^d(j)}{\theta^c(j)} > \frac{\theta^d(k)}{\theta^c(k)}$ with $j = 0, ..., k-1$. The second term is equal to zero for $k = 0$ and is strictly positive for $k = 1, ..., N-1$ since $\frac{\theta^d(j)}{\theta^c(j)} > \frac{\theta^d(k+1)}{\theta^c(k+1)}$ with $j = 0, ..., k-1$. Finally, the last term is always positive since $\frac{\theta^d(k)}{\theta^c(k)} > \frac{\theta^d(k+1)}{\theta^c(k+1)}$. It follows that (18) is strictly positive and therefore (17) holds. (17) implies that

$$\tfrac{\theta^d(0)}{\theta^c(0)} > \tfrac{\theta^d(0)+\theta^d(1)\eta(1)}{\theta^c(0)+\theta^c(1)\eta(1)} > \tfrac{\theta^d(0)+\theta^d(1)+\theta^d(2)\eta(2)}{\theta^c(0)+\theta^c(1)+\theta^c(2)\eta(2)} > ... > \tfrac{\theta^d(0)+\theta^d(1)+...+\theta^d(N)\eta(N)}{\theta^c(0)+\theta^c(1)+...+\theta^c(N)\eta(N)}. \tag{19}$$

Thus, $\mathcal{L}$ reaches its maximum value of $\frac{\theta^d(0)}{\theta^c(0)}$ when $\eta(0) > 0$ and $\eta(1) = ... = \eta(N) = 0$. The next step is to check the IC constraint for this punishment scheme. If the IC constraint is satisfied then this is the optimal punishment scheme and $\eta(0)$ is characterized by the binding IC constraint. If the IC constraint does not hold then one has to consider the second largest value of $\mathcal{L}$ from ranking (19), $\frac{\theta^d(0)+\theta^d(1)\eta(1)}{\theta^c(0)+\theta^c(1)\eta(1)}$, which is reached when $\eta(0) = 1$, $\eta(1) > 0$ and $\eta(2) = ... = \eta(N) = 0$. If the IC constraint is satisfied for $\eta(0) = 1$, $\eta(1) > 0$ and $\eta(2) = ... = \eta(N) = 0$, then this is the optimal punishment scheme and $\eta(1)$ is characterized by the binding IC constraint. Otherwise, one has to consider the third largest value of $\mathcal{L}$ from ranking (19), check the IC constraint for the punishment scheme corresponding to this value of $\mathcal{L}$, and continue this process until the IC constraint is satisfied. *Q.E.D.*

Proof of Proposition 5

Denote $R(N,q) \equiv \frac{q(N-1)}{\frac{1-\delta}{\delta(1-q)^{N-1}}+1}$. Suppose that $\tilde{k}=0$ characterizes the optimal punishment scheme for some $N_1 \geq 2$. Then $c-1 < R(N_1,q)$. Consider an increase in the group size from N_1 to N_1+1. $\tilde{k}=0$ will still characterize the optimal punishment scheme for N_1+1 if $R(N_1,q) < R(N_1+1,q)$, which amounts to

$$\frac{(1-q)^{N_1}}{N_1 q - 1} > \frac{1-\delta}{\delta}.$$

Assume that the above ineqality holds for some N_1 and consider the impact of a group-size increase from N_1 to N_1+1 on $\eta(0)$. It is easy to show that $\eta^{N_1}(0) \gtreqless \eta^{N_1+1}(0)$ if and only if $N_1 \lesseqgtr \frac{c}{q}$.

When $\tilde{k}=0$, IC condition holds if and only if $c-1 < R(N,q)$. Suppose that $c-1 < R(N_1,q)$ holds for some N_1. Consider whether the IC constraint is still satisfied when the group size increases from N_1 to N_1+1. The group-size effect is positive if $R(N_1+1,q) > R(N_1,q)$ and negative if $R(N_1+1,q) < R(N_1,q)$. Let $\tilde{k}$ still be set at 0. Use $R(N,q)$, we can obtain $R(N_1+1,q) = \frac{qN}{\frac{1-\delta}{\delta(1-q)^N}+1}$. Simple algebra shows the following:

$$R(N_1,q) \lesseqgtr R(N_1+1,q) \ \Leftrightarrow \ \frac{(1-q)^{N_1}}{N_1 q - 1} \gtrless \frac{1-\delta}{\delta}.$$

Note that $\frac{(1-q)^{N_1}}{N_1 q-1}$ is a strictly decreasing function of N_1. Moreover, when $N \to \infty$, $\frac{(1-q)^{N_1}}{N_1 q-1}$ tends to zero and the $\tilde{k}=0$ cannot be the solution for group size N_1+1. This implies that, for $\tilde{k}=0$ to be the solution for group size N_1+1, it is necessary and sufficient to have $\frac{(1-q)^2}{2q-1} > \frac{1-\delta}{\delta}$, or equivalently, $\frac{1}{2} < q < \sqrt{\frac{1-\delta}{\delta}} + \frac{1}{\delta}$. However, since $\sqrt{\frac{1-\delta}{\delta}} + \frac{1}{\delta} > 1$, the requirement reduces to $\frac{1}{2} < q$. Then, if $\frac{1}{2} < q$, $R(N_1,q) < R(N_1+1,q)$ when N_1 is relatively small and $R(N_1,q) > R(N_1+1,q)$ when N_1 is relatively big.

Now suppose that both $R(N_1,q)$ and $R(N_1+1,q)$ are strictly larger than $c-1$ so that $\tilde{k}=0$ is the solution for both group size N_1 and N_1+1. We want to see how $\eta(0)$ corresponds to the increase of N. Equation (**??**) gives the expression of $\eta(0)$

as a function of N. Define $R_0(N,q) = \frac{(1-\delta)(c-1)}{\delta[q(N-1)-(c-1)](1-q)^{N-1}}$. The group-size effect is positive if $R_0(N_1+1,q) < R_0(N_1,q)$ and negative if $R_0(N_1+1,q) > R_0(N_1,q)$. Simple algebra delivers the following:

$$R_0(N_1,q) \lesseqgtr R_0(N_1+1,q) \iff N_1 \gtrless \frac{c}{q}.$$

For every large N_1, $R_0(N_1,q) < R_0(N_1+1,q)$. To have $R_0(N_1,q) > R_0(N_1+1,q)$ for some N_1, it is necessary and sufficient to require $2 < \frac{c}{q}$. If this condition holds, $R_0(N_1,q) > R_0(N_1+1,q)$ when the group size is relatively small and $R_0(N_1,q) < R_0(N_1+1,q)$ when the group size is relatively large.

As a direct result of the analysis above, the group-size effect is positive when N_1 is relatively small and negative when N_1 is relatively large if and only if $\frac{1}{2} < q < \frac{c}{2}$. *Q.E.D.*