ON THE EQUIVALENCE BETWEEN (QUASI-)PERFECT AND SEQUENTIAL EQUILIBRIUM

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We prove the generic equivalence between quasi-perfect and sequential equilibrium. Combining this result with Blume and Zame (1994) shows that perfect, quasi-perfect and sequential equilibrium coincide in generic games.

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1. INTRODUCTION

BACKWARDS INDUCTION HAS been implemented in the literature through several equilibrium concepts for extensive-form games. *Extensive-form perfect equilibrium* (Selten, 1975), *sequential equilibrium* (Kreps and Wilson, 1982) and *quasi-perfect equilibrium* (van Damme, 1984) are (together with subgame perfection) the most prominent examples. Sequential equilibrium is the less demanding of these three concepts. Every extensive-form perfect as well as every quasi-perfect equilibrium is sequential. In turn, Blume and Zame (1994) show that for generic extensive-form games every sequential equilibrium is also extensive-form perfect (henceforth simply *perfect*).

Nevertheless, there is no inclusion relationship between quasi-perfect and perfect equilibrium. Indeed, Mertens (1995) gives an example of a game where quasi-perfect and perfect equilibrium select disjoint sets of strategy profiles. As Mertens argues, since quasi-perfect equilibria are normal form perfect—which can be understood as a strong version of admissibility—it seems that quasi-perfect equilibrium is superior to the perfect equilibrium concept. In fact, more recently Govindan and Wilson (2006, 2012) use quasi-perfect equilibrium as one of their building blocks to axiomatize and characterize strategic stability.

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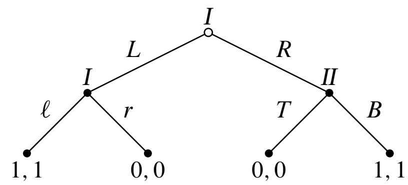

FIGURE 1. (R, ℓ, B) is perfect but not quasi-perfect.

A standard example that is used to show that perfect equilibrium may select unreasonable equilibria is depicted in Figure 1 (this is Example 4 in van Damme, 1984). The strategy profile (R, ℓ, B) is a sequential and a perfect equilibrium. But it is not an admissible strategy profile and consequently not a quasi-perfect equilibrium. The current paper shows that this example is exceptional in the space of games with that extensive form. More precisely, we prove that for generic extensive form games the sets of sequential and quasi-perfect equilibria coincide. This, together with the aforementioned result by Blume and Zame, implies that for generic extensive form games the sets of perfect, sequential and quasi-perfect equilibria are the same.

We follow Blume and Zame (1994) very closely. In Section 2 we introduce notation and terminology for extensive form games. Section 3 defines quasi-perfect equilibria as limit points of sequences of ε-quasi-perfect equilibria. Instead of providing the usual definition of sequential equilibrium as sequentially rational consistent assessments, we give a characterization of sequential equilibrium strategies based on ε-quasi-perfect equilibria of nearby games. This allows a simple comparison between quasi-perfect and sequential equilibrium that leads to proving the generic equivalence result in Section 4.

2. Preliminaries

In this section we introduce notation and definitions for finite extensive games with perfect recall.

An *extensive-form* is a tuple $\Gamma = (\mathcal{N}, T, \leq, P, H, C, \rho)$. The set of players is $\mathcal{N} = \{1, \ldots, N\}$. Players are indexed by $n = 1, \ldots, N$ and, as usual, the symbol $-n$ is used to denote $\mathcal{N} \smallsetminus \{n\}$.

The finite set of nodes T partially ordered by $\leq$ and contains the set of decision nodes X and the set of final nodes Z. The set of decision nodes X is partitioned by the *player partition* $P = (P_0, P_1, \ldots, P_N)$, where P_n represents the set of nodes where player n has to move (P_0 corresponds to the set of nodes where Nature moves). The *information partition* $H = (H_1, \ldots, H_N)$ contains the information structure of the extensive form, where for each n, the collection H_n partitions P_n into *information sets* $h \in H_n$. The set of *choices* in the extensive form is C and $C(h)$ will denote the set of choices available at the information set h. Finally, ρ specifies the probability distributions over the moves of Nature.

An extensive-form game $\Gamma(u)$ is obtained from the extensive-form Γ by specifying for each player n a Bernullian payoff function $u_n\colon Z \to \mathbb{R}$. Therefore, $U_n = \mathbb{R}^Z$ is the space of player n's payoffs and $U = \prod_n U_n = (\mathbb{R}^Z)^N$ is the space of games with extensive-form Γ.

Kuhn's Theorem (Kuhn, 1953) allows us to focus on behavior strategies. A *behavior strategy* s_n of player n specifies for each information set $h \in H_n$ where she has to move a probability distribution $s_n(\cdot \mid h)$ over the set of choices $C(h)$. The set of behavior strategies for player n is S_n and its (finite) subset of pure strategies is $I_n \subset S_n$. Furthermore, the set of completely mixed behavior strategies for player n is S_n°. We also write $S = \prod_n S_n$, $S_{-n} = \prod_{m \neq n} S_m$, $S^\circ = \prod_n S_n^\circ$ and $S_{-n}^\circ = \prod_{m \neq n} S_m^\circ$ for the corresponding sets of strategy profiles.

A strategy profile $s \in S$ induces (together with ρ) a probability distribution on the set of final nodes Z. Let $\mathbb{P}[z \mid s]$ be the probability that $z \in Z$ is reached if the strategy profile s is played. The expected utility to player n if $s = (s_{-n}, s_n)$ is played and the utility vector is $u \in U$ is given by

$$v_n(s_{-n}, s_n, u) = \sum_{z \in Z} u_n(z)\ \mathbb{P}[z \mid s_{-n}, s_n].$$

We also need to define the expected utility that player n obtains once each one of her information sets $h \in H_n$ is reached. These expected utilities depend on the conditional probability induced on Z by the strategy profile once the information set has been reached. However, some information sets may be reached with probability zero under some strategy profiles. Thus, we can only define these expected utilities for those strategy profiles for which they are well defined. To this end, for an information set $h \in H_n$ of player n, let $I(h)$ and $S(h)$ define the sets of pure and mixed strategy profiles that induce a play of the game that reaches a node in h. Note, in particular, that $S^\circ \subset S(h)$ for every h. If $I_n(h)$, $I_{-n}(h)$, $S_n(h)$ and $S_{-n}(h)$ are the corresponding projections of $I(h)$ and $S(h)$ on S_n and S_{-n}, perfect recall implies that $I(h) = I_n(h) \times I_{-n}(h)$ and $S(h) = S_n(h) \times S_{-n}(h)$. Moreover, let $Z(h)$ denote the final nodes that come after some node in h.

The expected utility to player n at the information set h when the strategy profile $s = (s_{-n}, s_n) \in S(h)$ is played is given by:

$$v_n^h(s_{-n}, s_n, u) = \sum_{z \in Z(h)} u_n(z)\ \frac{\mathbb{P}[z \mid s_{-n}, s_n]}{\mathbb{P}[Z(h) \mid s_{-n}, s_n]},$$

where the probability in the denominator is computed in the usual manner.

3. Sequential and quasi-perfect equilibrium

Before we define quasi-perfect and sequential equilibrium we need some additional notation. If $h \in H_n$ and $c \in C(h)$, we denote as $I_n(h, c)$ the subset of strategies in $I_n(h)$ that prescribe action c at h. Furthermore, if $c \in C(h)$ and $h \in H_n$ we use the substitution notation $s_n|_h c$ to denote the strategy of player n that prescribes the same behavior as s_n at every information set but h, where it assigns probability one to choice c.

We define quasi-perfect equilibrium using ε-quasi-perfect equilibria:

▶ Definition 1. A completely mixed strategy profile $s^\varepsilon \in S^\circ$ is an *ε-quasi-perfect equilibrium* of the game $\Gamma(u)$ if for every player $n \in \mathcal{N}$, every information set $h \in H_n$, and every two choices $c, c' \in C(h)$ the following holds

$$\max_{i_n \in I_n(h,c)} v_n^h(s_{-n}^\varepsilon, i_n, u) < \max_{j_n \in I_n(h,c')} v_n^h(s_{-n}^\varepsilon, j_n, u) \text{ implies } s_n^\varepsilon(c \mid h) \leq \varepsilon.$$

▶ Definition 2. A strategy profile $s \in S$ is a *quasi-perfect equilibrium* of the game $\Gamma(u)$ if it is the limit point as ε goes to zero of ε-quasi-perfect equilibria.

Furthermore, we let $QE\colon U \twoheadrightarrow S$ denote the quasi-perfect equilibrium correspondence.

We move to define sequential equilibrium. A sequential equilibrium is a strategy profile *and* a sequence of beliefs. To compare quasi-perfect and sequential equilibria we need to focus on sequential equilibrium strategies. Our starting point is the following useful characterization of sequential equilibrium strategies in terms of sequences of strategy profiles and sequences of payoffs (Kreps and Wilson, 1982, Proposition 6):

Proposition 3 *A strategy profile $s \in S$ is a sequential equilibrium strategy of the game $\Gamma(u)$ if and only if there is a sequence of completely mixed strategy profiles $\{s^t\}_{t=1}^\infty \subset S^\circ$ and a sequence of payoff functions $\{u^t\}_{t=1}^\infty \subset U$ such that:*[1]

- *$\{s^t\}_{t=1}^\infty$ converges to s, $\{u^t\}_{t=1}^\infty$ converges to u; and*
- *for every index t, every player n, every information set $h \in H_n$ and every two choices $c, c' \in C(h)$, if $s_n(c \mid h) > 0$, then*

$$v_n^h(s_{-n}^t, s_n^t|_h c, u^t) \geq v_n^h(s_{-n}^t, s_n^t|_h c', u^t).$$

(From now on we use the term sequential equilibrium referring only to the strategy component.) We are interested in a similar characterization of sequential equilibrium that uses the ε-quasi-perfect equilibrium conditions. The following proposition serves this purpose:

Proposition 4 *A strategy profile $s \in S$ is a sequential equilibrium of $\Gamma(u)$ if and only if there is a sequence $\{\varepsilon^t\}_{t=1}^\infty \subset (0,1]$, a sequence of completely mixed strategy profiles $\{s^t\}_{t=1}^\infty \subset S^\circ$ and a sequence of payoff functions $\{\tilde{u}^t\}_{t=0}^\infty \subset U$ such that:*

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[1] A perfect equilibrium of $\Gamma(u)$ is defined similarly. We only need to restrict the sequence of payoff functions $\{u^t\}_{t=1}^\infty$ so that $u^t = u$ for all t.

- *$\{\varepsilon^t\}_{t=1}^{\infty}$ converges to 0, $\{s^t\}_{t=1}^{\infty}$ converges to s, $\{\widetilde{u}^t\}_{t=1}^{\infty}$ converges to u, and*
- *for every index t, s^t is an ε^t-quasi-perfect equilibrium of $\Gamma(\widetilde{u}^t)$.*

Proof. See Appendix A. □

Henceforth, we let $SE\colon U \twoheadrightarrow S$ represent the sequential equilibrium correspondence.

Proposition 4 characterizes the sequential equilibria of a game as the set of limit points of ε-quasi-perfect equilibria of nearby games. Intuitively, if a strategy profile s is a sequential equilibrium of $\Gamma(u)$ then Proposition 3 implies that it can be approximated by a sequence of equilibria of nearby payoff-perturbed games (Blume and Zame, 1994, Proposition B). Of course, this does not imply that the equilibria of the perturbed games be also ε-quasi-perfect equilibria for some ε. However, player's payoff vectors can be varied slightly to make it so. To conclude, one can show that these variations in the payoffs vanish as the sequence of games approaches the true game.

4. The result

Following Blume and Zame (1994), we exploit the semi-algebraic structure of most game theoretical constructions. A set is semi-algebraic if it can be defined by a finite system of polynomial equalities and inequalities. A correspondence (function) is semi-algebraic if its graph is a semi-algebraic set. The Tarski-Seidenberg Theorem (Tarski, 1951; Seidenberg, 1954) guarantees that every first-order formula (a expression involving constants, variables, the universal and existential quantifiers and the standard algebraic operations) defines a semi-algebraic set. Blume and Zame (1994) use the Tarski-Seidenberg Theorem to show that the Nash, perfect and sequential equilibrium correspondences are semi-algebraic. As they suggest, their argument can be extended to establish the semi-algebraic nature of many equilibrium refinements. One can easily apply it here to show that the quasi-perfect equilibrium correspondence is semi-algebraic. In fact, every set and correspondence that we consider in this paper is semi-algebraic.[2]

The basic result on semi-algebraic correspondences that we use is stated in Lemma 6 below and proved in Blume and Zame (1994). But before that we need to introduce the usual sequential characterizations of continuity for correspondences.

▶ Definition 5. Let $F\colon X \twoheadrightarrow Y$ be a compact-valued correspondence.

- F is upper-hemicontinuous at x if and only if for every sequence $\{x^t\}_{t=1}^{\infty} \subset X$ converging to x, the limit y of any sequence $\{y^t\}_{t=1}^{\infty} \subset Y$ such that $y^t \in F(x^t)$ for all t satisfies $y \in F(x)$.

[2] For a detailed exposition of semi-algebraic theory the reader is referred to Bochnak et al. (1998).

- F is lower-hemicontinuous at x if and only if for every sequence $\{x^t\}_{t=1}^{\infty} \subset X$ converging to x and for every $y \in F(x)$ there exist a subsequence $\{x^{t_k}\}_{k=1}^{\infty}$ and a sequence $\{y^{t_k}\}_{k=1}^{\infty}$ that converges to y such that $y^{t_k} \in F(x^{t_k})$ for all k.

- F is continuous at x if and only if it is both upper-hemicontinuous and lower-hemicontinuous at x.

We can now turn to the announced result on semi-algebraic correspondences:

Lemma 6 *Let $F\colon X \twoheadrightarrow Y$ be a compact-valued and semi-algebraic correspondence. Then F is continuous at every point of the complement of a (relatively) closed, lower-dimensional, semi-algebraic subset of X.*

As the perfect, quasi-perfect and sequential equilibrium correspondences are compact-valued and semi-algebraic, Lemma 6 has important consequences to study their continuity points.

Fix some $\bar{\varepsilon} > 0$, the ε-quasi-perfect equilibrium correspondence is denoted by $\varphi\colon U \times (0,\bar{\varepsilon}] \twoheadrightarrow S^{\circ}$. Let $W = \mathrm{Graph}(\varphi) \subset U \times (0,\bar{\varepsilon}] \times S^{\circ}$. The strategy profile s is a sequential equilibrium of $\Gamma(u)$ if and only if there is a sequence $\{(u^t,\varepsilon^t,s^t)\}_{t=1}^{\infty} \subset W$ converging to $(u,0,s)$. If $\mathrm{cl}(W)$ is the closure of W and $\mathrm{cl}(W)_u = \{(\varepsilon,s) : (u,\varepsilon,s) \in \mathrm{cl}(W)\}$ we can say that s is a sequential equilibrium of $\Gamma(u)$ if and only if $(0,s) \in \mathrm{cl}(W)_u$. Likewise, the strategy profile s is a quasi-perfect equilibrium of $\Gamma(u)$ if and only if $(u,0,s)$ is the limit point of some sequence $\{(u,\varepsilon^t,s^t)\}_{t=1}^{\infty} \subset W$. If we let $W_u = \{(\varepsilon,s) : (u,\varepsilon,s) \in W\}$ we can say that s is a quasi-perfect equilibrium of $\Gamma(u)$ if and only if $(0,s) \in \mathrm{cl}(W_u)$. Additionally, we define the correspondence $\psi\colon U \twoheadrightarrow [0,\bar{\varepsilon}] \times S$ by $\psi(u) = \mathrm{cl}(W_u)$.

We begin by characterizing the set of games for which quasi-perfect and sequential equilibria coincide using the upper-hemicontinuity points of ψ.

Proposition 7 *The sets of quasi-perfect and sequential equilibria coincide at u if and only if ψ is upper-hemicontinuous at u.*

Proof. Let ψ be upper-hemicontinuous at u and let s be a sequential equilibrium of $\Gamma(u)$. There is a sequence $\{(u^t,\varepsilon^t,s^t)\}_{t=0}^{\infty} \subset W$ converging to $(u,0,s)$. Along this sequence $(\varepsilon^t,s^t) \in \psi(u^t)$ for every t. Upper-hemicontinuity of ψ at u implies that $(0,s) \in \psi(u)$ which in turn implies that s is a quasi-perfect equilibrium of $\Gamma(u)$.

Suppose now that $QE(u) = SE(u)$. The correspondence $\bar{\psi}\colon U \twoheadrightarrow [0,\bar{\varepsilon}] \times S$ given by $\bar{\psi}(u) = \mathrm{cl}(W)_u$ has a closed graph and, therefore, is upper-hemicontinuous everywhere. Furthermore, the graphs of the correspondences $\bar{\psi}$ and ψ can only differ at those points where $\varepsilon = 0$. That is, $QE(u) = SE(u)$ implies $\bar{\psi}(u) = \psi(u)$, from where we can conclude that ψ is upper-hemicontinuous at u. □

The correspondence ψ is semi-algebraic.[3] From Lemma 6 it follows that ψ is upper-hemicontinuous at every point of the complement of a closed lower-dimensional semi-algebraic set. Therefore, Proposition 7 implies the generic equivalence between sequential and quasi-perfect equilibrium. Moreover, the analogous result involving perfect and sequential equilibrium has been established by Blume and Zame (1994, Theorem 4).[4] Hence, we obtain:

Theorem 8 *There is a closed, lower-dimensional semi-algebraic subset $U_0 \subset U$ such that for every $u \in U \setminus U_0$ the sets of perfect, quasi-perfect and sequential equilibria coincide.*

Since quasi-perfect equilibria are always normal-form perfect we also obtain that in the complement of a closed, lower-dimensional semi-algebraic subset of payoffs every extensive-form perfect equilibrium is also normal-form perfect.[5]

Analogously to Blume and Zame (1994) we also obtain:

Corollary 9 *The quasi-perfect and sequential equilibrium correspondences coincide at every point where the first correspondence is upper-hemicontinuous and the second correspondence lower-hemicontinuous.*

Proof. Let SE be lower-hemicontinuous at u and let $s \in SE(u)$. Take any sequence $\{u^t\}_{t=1}^{\infty}$ converging to u and such that $\{u^t\}_{t=1}^{\infty} \subset U \setminus U_0$. Lower hemicontinuity of SE implies that (passing to a subsequence if necessary) we can find $\{s^t\}_{t=1}^{\infty}$ converging to s such that $s^t \in SE(u^t)$ for all t. Since $QE(u^t) = SE(u^t)$ also holds for all t we actually have a sequence of quasi-perfect equilibria converging to s. Upper-hemicontinuity of QE implies that s is a quasi-perfect equilibrium of $\Gamma(u)$. □

Appendix A. Proof of Proposition 4

The starting point of the proof is Proposition 3. Thus, before proving the result, we provide a definition of perfect equilibrium based on perturbed games.

A *perturbation* for the extensive form is a function $\eta\colon C \to \mathbb{R}_{++}$ such that $\sum_{c \in C(h)} \eta(c) < 1$ for every information set h. Given a perturbation η the set of perturbed strategies of player n is

$$S_n(\eta) = \{s_n \in S_n : s_n(c \mid h) \geq \eta(c) \text{ for all } c \in C(h), h \in H_n\}.$$

The *perturbed game* $\Gamma(u, \eta)$ is the extensive-form game with payoffs u and players are constrained to play strategy profiles in $S(\eta) = \prod_n S_n(\eta)$.

[3] Notice that $\text{Graph}(\psi) = W \cup \{(u, 0, s) : (u, s) \in \text{Graph}(QE)\}$ and that both sets in the union are semi-algebraic.

[4] In fact, by letting W denote the graph of the ε-perfect equilibrium correspondence we also provide an alternative proof to the generic equivalence between sequential and perfect equilibrium.

[5] Reny (1992) shows that generically, weakly sequential equilibria are normal-form perfect. Since all sequential equilibria, hence all extensive-form perfect equilibria, are weakly sequential, it can be seen that generically, extensive-form perfect equilibria are normal-form perfect.

▶ DEFINITION A.1. A strategy profile $s \in S$ is a perfect equilibrium if there is a sequence of perturbations $\{\eta^t\}_{t=0}^{\infty}$ converging to zero and a sequence of strategy profiles $\{s^t\}_{t=0}^{\infty}$ converging to s such that s^t is a Nash equilibrium of $\Gamma(u, \eta^t)$ for every t.

Of course, this definition is equivalent to the one indicated in footnote 1.

Proof of the "only if" part of Proposition 4. Take a sequential equilibrium s of $\Gamma(u)$. By Proposition 3 we know that there is a sequence $\{(u^t, \eta^t, s^t)\}_{t=1}^{\infty}$ converging to $(u, 0, s)$ such that s^t is an equilibrium of the perturbed game $\Gamma(u^t, \eta^t)$ for all t. For the time being, fix a member (u^t, η^t, s^t) of the sequence. For any $h \in H_n$ and $c \in C(h)$ construct the set:

$$P_n^t(h, c) = \left\{i_n \in I_n(h, c) : i_n(c' \mid h') = 1 \text{ and } h < h' \text{ imply } s_n^t(c' \mid h') > \eta(c')\right\}.$$

Furthermore, let $Q_n^t(h, c) = I_n(h, c) \setminus P_n^t(h, c)$.

We define a probability measure on the set of pure continuation strategies. Let $\overline{\mathbb{P}}[i_n \mid h, s_n]$ be the probability that the strategy s_n assigns to the set of pure strategies that coincide with i_n at every information set that follows h (including h itself). Formally, let $C_n(h, i_n) = \left\{c \in C : i_n(c \mid h') = 1 \text{ and } h \le h'\right\}$ then $\overline{\mathbb{P}}[i_n \mid h, s_n] = \prod_{c \in C_n(h,i_n)} s_n(c \mid h)$. In particular, note that

$$\sum_{i_n \in I_n(h,c)} \overline{\mathbb{P}}[i_n \mid h, s_n|_h c] = 1, \quad \text{for all } s_n.$$

Therefore, if the choice $d \in C(h)$ is such that $s_n^t(d \mid h) > \eta^t(d)$ we can write:

$$v_n^h(s_{-n}^t, s_n^t|_h d, u^t) = \sum_{i_n \in P_n^t(h,d)} \overline{\mathbb{P}}[i_n \mid h, s_n^t|_h d] v_n^h(s_{-n}^t, j_n, u^t) + \sum_{j_n \in Q_n^t(h,d)} \overline{\mathbb{P}}[j_n \mid h, s_n^t|_h d] v_n^h(s_{-n}^t, j_n, u^t).$$

Since s^t is an equilibrium of $\Gamma(u^t, \eta^t)$ the value of the function $v_n^h(s_{-n}^t, i_n, u^t)$ is the same for every $i_n \in P_n^t(h, c)$. Take an arbitrary $i_n^d \in P_n^t(h, d)$ and rewrite the previous expression

$$v_n^h(s_{-n}^t, s_n^t|_h d, u^t) = v_n^h(s_{-n}^t, i_n^d, u^t) - l_n^t(h, d), \tag{A.1}$$

where the last terms equals

$$l_n^t(h, d) = \sum_{i_n \in Q_n^t(h,d)} \overline{\mathbb{P}}[i_n \mid h, s_n^t|_h d] \left(v_n^h(s_{-n}^t, i_n^d, u^t) - v_n^h(s_{-n}^t, i_n, u^t)\right).$$

Consider now a pure strategy $j_n \in I(h)$ that maximizes $v_n^h(s_{-n}^t, i_n, u^t)$ over $I(h)$. Let $s_n^{t,j_n} \in S_n(\eta^t)$ be the perturbed strategy that is located in the vertex of $S_n(\eta^t)$ which is closest to j_n. Then, $\overline{\mathbb{P}}[i_n \mid h, s_n^{t,j_n}]$ is smaller than $\varepsilon^t = \max_c \{\eta^t(c)\}$ for every pure strategy in $I_n(h)$ that is not j_n.

We can write

$$v_n^h(s_{-n}^t, s_n^{t,j_n}, u^t) = v_n^h(s_{-n}^t, j_n, u^t) - L_n^t(h), \tag{A.2}$$

where the last terms equals

$$L_n^t(h) = \sum_{i_n \in I_n(h) \setminus \{j_n\}} \overline{\mathbb{P}}[\, i_n \mid h, s_n^{t,j_n}] \left(v_n^h(s_{-n}^t, j_n, u^t) - v_n^h(s_{-n}^t, i_n, u^t) \right).$$

We have the following inequalities:

$$v_n^h(s_{-n}^t, s_n^t|_h d, u^t) \geq v_n^h(s_{-n}^t, s_n^t, u^t) \geq v_n^h(s_{-n}^t, s_n^{t,j_n}, u^t),$$

where the first inequality follows because in the the perturbed game $\Gamma(u^t, \eta^t)$ choice d is optimal for n's agent at the information set h and the second inequality follows because s^t is an equilibrium of such a perturbed game.

Combining the last sequence of inequalities with (A.1) and (A.2) we obtain:

$$v_n^h(s_{-n}^t, i_n^d, u^t) + \left(L_n^t(h) - l_n^t(h, d) \right) \geq v_n^h(s_{-n}^t, j_n, u^t).$$

To sum up, a strategy of player n that prescribes action d at the information set h is optimal in the perturbed game $\Gamma(u^t, \eta^t)$. However, the strategy j_n maximizes player n's utility at h when she does not consider her mistakes in the future. We are going to use the last inequality to construct a payoff $\widetilde{u}^t$ such that s^t is an ε^t-quasi-perfect equilibrium of $\Gamma(\widetilde{u}^t)$. We will later show that the new sequence $\{\widetilde{u}^t\}_{t=1}^{\infty}$ converges to u.

Start with an information set $h \in H_n$ with no preceding information set in H_n. The set $Z(h, i_n^d) \subset Z(h)$ is the set of final nodes that come after h and after all the choices prescribed by i_n^d. Add $L_n^t(h) - l_n^t(h, d)$ to the utility that player n obtains from each $z \in Z(h, i_n^d)$.

Consider now an information set $h' \in H_n$ that follows h immediately. Let d', $i_n^{d'}$, j_n', $L_n^t(h')$, and $l_n^t(h', d')$ be constructed as before and add $L_n^t(h') - l_n^t(h', d')$ to the utility that player n obtains from each $z \in Z(h', i_n^{d'})$. To guarantee that player n's optimality conditions are not affected at h, also add this perturbation to player n's utilities to all final nodes $z \in Z(h, i_n)$. Continue with this procedure with each subsequent information set and for each player. Since the game is finite, the procedure ends after a finite number of steps and we obtain a game $\Gamma(\widetilde{u}^t)$ such that s^t is an ε^t-quasi-perfect equilibrium of $\Gamma(\widetilde{u}^t)$ (with $\varepsilon^t = \max_{c \in C} \{\eta^t(c)\}$). We can check that the sequence of numbers $\{L_n^t(h) - l_n^t(h, d)\}_{t=1}^{\infty}$ converges to zero. This proves the result.

Proof of the "if" part of Proposition 4. Take a sequence $\{(\widetilde{u}^t, \varepsilon^t, s^t)\}_{t=1}^{\infty}$ converging to $(u, 0, s)$ where, for each t, the strategy s^t is an ε^t-quasi-perfect equilibrium of $\Gamma(\widetilde{u}^t)$. We have to show that for each t we can find a new payoff vector u^t and a perturbation η^t such that s^t is an equilibrium of the perturbed game $\Gamma(u^t, \eta^t)$ and that, furthermore, $\{u^t\}_{t=1}^{\infty}$ converges to u.

By letting $\eta^t(c) = \min\{s^t(c \mid h), \varepsilon^t\}$ we construct the vector of perturbations. The construction of the payoff vector for each t follows analogous lines to the proof of the *only if* part of Proposition 4 and, hence, it is omitted. □

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